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DIFFRACTION, REFRACTION, AND REFLECTION OF RADIO WAVES

THIRTEEN PAPERS BY

V. A. FOCK

INTRODUCTION BY V. I. SMIRNOV

APPENDIX BY M. A. LEONTOVICH

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EDITORS' NOTE

The Soviet physicist V.A. Fock is well known by physicists for his work in quantum mechanics, particularly in connection with the Hartree-Fock theory of self-consistent fields. The purpose of this collection is to acquaint the reader with Fock's more recent work on the propagation, refraction, and diffraction of radiowaves. Fock's early papers on this subject (the first five papers in this collection) appeared in English almost a decade ago. However, all of his more recent work has been published in Russian and is relatively unknown outside the Soviet Union.

The translations in this collection have been based upon translations obtained from several sources. Mr. Herman V. Cottony of the National Bureau of Standards and Miss A. Pingell of the Naval Research Laboratory, respectively, made the original translations of Chapters VI and XI of this collection. The translator of Chapter VIII is unknown to the editors. The remainding chapters were translated by Morris D. Friedman. Chapters VIII, IX, and X were made by Morris D. Friedman, Inc., Newtonville, Massachusetts. Chapters XII and XIII were made in cooperation with Lincoln Laboratory.

According to the Library of Congress scheme for the transliteration of the Russian alphabet, Fock's name appears as Fok. However, because of the more general use in scientific literature of the form Fock the editors have retained this form in this collection.

N.A.L.
P.B., Jr.

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CONTENTS

CONTENTS (cont'd)

CONTENTS (cont'd)

CONTENTS (cont'd)

Abbreviations for Soviet Journals

IAN (Ser. Fiz.), Izvestiia Akademii Nauk SSSR - Seriia Fizicheskaia (Bulletin of the Academy of Sciences of USSR - Physics Series).

ZETF, Zhurnal Eksperimentalnoi i Theoreticheskoi Fiziki (Journal of Experimental and Theoretical Physics).

UFN, Uspekhi Fizicheskikh Nauk (Progress of Physical Sciences).

Radiotekh. i Elektr., Radiotekhnika i Elektronika (Radioengineering and Electronics)

V.A. FOCK'S CONTRIBUTIONS TO DIFFRACTION THEORY

V.I. Smirnov

1. INTRODUCTION

V.A. Fock became interested in diffraction problems comparatively recently. Within a short time he succeeded in obtaining numerous results which are very important both in theoretical and in practical aspects. By forecasting the paths of further investigations in this field, they undoubtedly are epochal in diffraction theory.

The solution of the problems of electromagnetic wave diffraction consists of finding solutions of the Maxwell equations subject to specific initial and boundary conditions on the diffracting surface and radiation conditions at infinity. The initial conditions are often replaced by the requirement that the solution be sinusoidal in time. Fock devoted himself to an analysis of problems of the last kind. Prior to the Fock investigations in the theory of electromagnetic wave diffraction, only solutions for a small number of problems for obstacles of a specific shape were known, such as: the infinite wedge, cylinders - circular, elliptic and parabolic - and also for the sphere. In addition, the problem of diffraction from a paraboloid of revolution, solved by Fock himself in 1944, should be added to the above list.

The previous solutions of the problems mentioned above, which were represented by series or by integrals, were not very useful in the important practical case when the wavelength is small in comparison to the dimensions of the obstacle, and they should be considered as only the first step in solving the problem. The next step must be the derivation of formulas from which qualitative physical consequences can be obtained and which are, in addition, suitable for practical computations. Hence, one of the possible directions of work in diffraction theory was the development of a method of isolating the principal parts out of the complex formulas which constitute the exact solution of the problem. The Fock investigations were made in this direction when solving the problems of diffraction from a conducting sphere as well as from a paraboloid of revolution. Naturally, the method cited is applicable only in those few cases when an exact solution can be constructed successfully. Consequently, an urgent need existed for the creation of an approximate method of solving diffraction problems which, while being general, would lead to relatively simple formulas.

The fundamental works of Fock on diffraction are devoted to the construction of such an approximate method and to the solution of a number of practical important problems by using this method. Fock developed and used the parabolic equation method proposed by Leontovich. This permitted him to give not only new simplified derivations of results he had obtained earlier by other means but also to generalize them in various directions (to take the finite conductivity of the body into account; to determine the field close to the surface as well as on the surface itself; to take atmospheric inhomogeneities into account in the problem of diffraction of radiowaves around the earth's surface).

As is every approximate method of solving boundary value problems, the Fock method is based on the smallness of certain parameters encountered in the problem. The quantities which are usually small in the problems of radiowave diffraction are: $\frac{1}{|\eta|}$ and $\frac{\lambda}{R}$, where $\eta = \varepsilon + i\frac{4\pi\sigma}{\omega}$ is the complex dielectric constant of the diffracting body; λ is the wavelength of the incident wave; R is a quantity of the order of the radius of curvature of the surface of the body.

If $|\eta| = \infty$ (perfect conductor), then the field within the conductor is zero, i.e., it is known in advance. This circumstance permits the diffraction problem to be formulated only for the space outside the body, which leads to substantial simplification. The situation in the imperfect conductor case is similar if the inequalities $|\eta| \gg 1$ and $\frac{R}{\lambda}\sqrt{|\eta|} \gg 1$ are satisfied.

In this case, the field within the conductor appears to be vanishingly small everywhere except in a surface layer of thickness of order $\lambda/|\eta|$, where the influence of this layer can be taken into account by using boundary conditions for the external field

$$\frac{4\pi j_x}{c} = \sqrt{\eta}\,(E_x - n_x E_n) = n_y H_z - n_z H_y \quad \text{, etc.} \tag{1}$$

where j_x, j_y, j_z are the components of the current density; n_x, n_y, n_z are the unit vector components normal to the body surface. Acad. M.A. Leontovich first suggested the aforementioned conditions in a rather different form.

Consequently, the approximate formulation of the diffraction problem is thereby reduced to a problem involving the fields exterior to the body. A further essential simplification in problems of radio-wave diffraction from bodies of arbitrary shape results from the principle of the field being local in the half-shadow region.

If the electromagnetic field near the surface of a conducting body were to be determined successfully, and, therefore, the current distribution in the surface layer, then the solution of the diffraction problems would be attained by simple well-known formulas for the vector potential. The field in the illuminated region near the body is subject, with a high degree of accuracy, to the Fresnel laws of reflection, and, therefore, can be determined easily; the field decreases rapidly to zero in the shadow region.

Consequently, the unattainable link in the approximate solution of the diffraction problems is the transition region (half-shadow) located near the geometric shadow boundary and with the shape of a band of width $d = \sqrt[3]{\frac{\lambda}{\pi} R_o^2}$, where R_o is the radius of curvature of a normal section of the body in the incident plane.

Fock succeeded in showing that the electromagnetic field in the half-shadow region is, to the accuracy of quantities of the order of

$\sqrt[3]{\frac{\lambda}{\pi R_0}}$, of local character, i.e., it depends only on the values of the incident wave field in the neighborhood of the given point, on the geometric shape of the body near this point, and on the electric properties of the conductor.

After the principle of the local field had been established, there remained only to find the solution of the diffraction problem for a convex body of sufficiently general shape, and to derive the approximate formulas for the field on its surface. It is convenient to take the paraboloid of revolution as such a body. In solving the problem of plane wave diffraction from a paraboloid, V.A. Fock used separation of variables in parabolic coordinates. He constructed the exact solution in the form of integrals and performed the approximate calculation of these integrals under the assumption that $ka \gg 1$, where k is the wave number and a is a parameter of the paraboloid of revolution: $x^2 + y^2 - 2az - a^2 = 0$.

The characteristic direction of the work on diffraction explained above is sufficient to indicate the important principles of the methods developed. Basically, these methods reduce to the following:

Fock indicated an effective method of approximately evaluating infinite series and integrals (containing a large parameter) which represent the exact solutions of certain problems of electromagnetic wave diffraction. This method permitted him to develop, for example, a rigorous theory on radiowave diffraction around the earth's surface surrounded by a homogeneous atmosphere ("Diffraction of Radiowaves

Around the Earth's Surface", 1946)[1]. He was also the first to establish the very important principle of the local character of the electromagnetic field in the half-shadow region, using widely the Leontovich conditions[2] in the approximate formulation of radiowave diffraction problems.

This work afforded him the opportunity to construct an approximate, but yet sufficiently accurate for practical needs, theory of radiowave diffraction from conductors of arbitrary shape as well as a theory of radiowave propagation around the earth taking inhomogeneities of the atmosphere into account. The explanation of this theory is given in "Theory of Radiowave Propagation in an Inhomogeneous Atmosphere for a Raised Source", (1950)[3].

These works on diffraction have played a very important part in the history of this question and, at the present time are among the clearest attainments in diffraction theory and its applications. Let us turn to a more detailed explanation of some of these works.

The problem of radiowave diffraction in a vacuum relative to a conducting sphere is solved in "Diffraction of Radiowaves Around the Earth's Surface"[4].

Let the sphere be of radius a and be characterized by the dielectric constant ε , the conductivity σ and the magnetic permeability unity. Let the spherical coordinates (r,θ,φ) be introduced and let a vertical electric dipole be placed at the point $r = b$, $\theta = 0$, where $b > a$. The electromagnetic field excited by such a dipole can be expressed by means of the Hertz function $U(r,\theta,\varphi)$ which satisfies the equation

(2) $$\Delta U + k^2 U = 0$$

Hence, in order to determine the value of the field on the sphere's surface, it is sufficient to know the quantities:

(3) $$U_a = U(a,\theta,\varphi) \quad \text{and} \quad U_a' = \left.\frac{\partial(rU)}{\partial r}\right|_{r=a} .$$

In 1908, Mie obtained an analytical representation for the function U as an infinite series of spherical functions. The extremely poor convergence of the series prevented qualitative physical consequences from being obtained and prevented practical use of the aforementioned exact solution of the problem. A major step toward a practical use of these series was made by Watson in 1918. But the transformed form of the solution was still unsatisfactory, both because of its complexity and because it was only applicable in the geometric shadow region (i.e., far from the horizon). Only in 1945 did Fock succeed in obtaining an expression for the Hertz function suitable for all cases.

Fock transforms the series for U_a and U_a' into complex integrals. But, in contrast to the preceding authors who tended to reduce the integrals to a sum of residues, Fock isolated from the integrals a principal term which yields sufficiently exact values for the functions investigated.

It was shown in this work that if waves passing through the thickness of the earth and waves circumscribing the earth because of diffraction are neglected because of their smallness, then the value of U_a can be represented by the following integral

(4) $$U_a = \frac{2e^{i\frac{3\pi}{4}}}{\pi\, kab\sqrt{2\sin\theta}} \int_C \nu\varphi(\nu) e^{i\nu\theta} G_\nu^* \, d\nu \;;$$ where

(5) $$\varphi(\nu) = \frac{\zeta_{\nu-\frac{1}{2}}(Kb)}{\zeta'_{\nu-\frac{1}{2}}(Ka) - \frac{K}{K_2}\chi_{\nu-\frac{1}{2}}(K_2 a)\zeta_{\nu-\frac{1}{2}}(Ka)} \;;$$

(6) $$G_\nu^* = \frac{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\nu+1)} F\left(\frac{1}{2}, \frac{1}{2}, \nu+1, \frac{1}{2} - \frac{i}{2} ctg\,\theta\right);$$

(7) $$\zeta_n(\rho) = \sqrt{\frac{\pi\rho}{2}}\, H^{(1)}_{n+\frac{1}{2}}(\rho); \quad \psi_n(\rho) = \sqrt{\frac{\pi\rho}{2}}\, J_{n+\frac{1}{2}}(\rho);$$

(8) $$\chi_n(\rho) = \frac{\psi'_n(\rho)}{\psi_n(\rho)}; \quad k = \frac{2\pi}{\lambda}$$ (λ - is the wavelength)

(9) $$K_2 = K\sqrt{\eta}\,; \quad \eta = \varepsilon + i\,\frac{4\pi\sigma}{cK}\,;$$

$F(\alpha,\beta,\gamma;\, z)$ is the hypergeometric function; the contour C is a line intersecting the positive part of the real axis going downward (to the left of the poles of $\varphi(\nu)$).

A similar integral is obtained for U'_a. The essential feature of this method of approach is that the integrals obtained can be calculated easily and with great accuracy for any value of θ.

The characteristic parameter of the aforementioned integrals is the quantity $p = \sqrt[3]{\frac{ka}{3}} \cos \gamma$ where γ is the angle between the vertical at the observation point and the source direction. If $p \gg 1$ and the observer is in the line of sight region (more accurately: if $kh \cos\gamma \gg 1$, where h is the height of the source above the earth), then the evaluation of the integrals leads to the well-known "reflection formula". This evaluation of the integrals leads to the Weyl-van der Pol formula valid for points at large distances from the source but still well within the line-of-sight.

The half-shadow region (where $p \approx 1$), for which approximate values of the field were not known) is of greatest interest. A method is indicated in this work of evaluating the integrals for this case and the following formula is obtained

$$(10) \qquad U_a = \frac{e^{ika\theta}}{a\theta} e^{-i\frac{\pi}{4}} \sqrt{\frac{x}{\pi}} \int_{\Gamma} e^{ixt} \frac{w_1(t-y)}{w_1'(t) - q w_1(t)} \, dt,$$

in which $w_1(t)$ is the complex Airey function related to the Hankel function of one third order by the relation

$$(11) \quad w_1(t) = \sqrt{\frac{\pi}{3}} e^{i\frac{2\pi}{3}} (-t)^{\frac{1}{2}} H_{1/3}^{(1)} \left[\frac{2}{3}(-t)^{3/2}\right].$$

The contour Γ goes from $i\infty$ to 0 and from 0 to $+\infty$;

$$(12) \qquad x = \left(\frac{ka}{2}\right)^{1/3} \theta; \quad y = \left(\frac{ka}{2}\right)^{-1/3} kh; \quad q = i\left(\frac{ka}{2}\right)^{1/3} \frac{\sqrt{\eta - 1}}{\eta}.$$

The formula for the half-shadow region is the main result of this work. It is applicable in all cases of practical interest. It transforms into the Weyl-van der Pol formula far from geometric shadow in the line-of-sight region. This formula can be reduced to a rapidly converging series when the transition is made into the shadow region where $(-p) \gg 1$.

In the work "Solution of the Problem of Propagation of Electromagnetic Waves Along the Earth's Surface by the Method of Parabolic Equations" (written jointly with M.A. Leontovich)[4], a problem is analyzed which is similar to the problem in the paper mentioned above but the method is essentially different.

The influence of the earth's surface is taken into account by the Leontovich approximate boundary conditions and terms in the field equations are neglected which are small and are of the order of $c \frac{1}{|\eta|}$ and $\frac{1}{ka}$. As a result, the "approximate" formulation of the problem for the spherical earth case is simplified substantially and is reduced to the problem of solving the parabolic equation

$$(13) \qquad \frac{\partial^2 w}{\partial y^2} + i\left[\left(x + \frac{y}{x}\right)\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\right] = 0 ,$$

in the region exterior to the earth and subject to the additional conditions

$$(14) \qquad \left.\frac{\partial w}{\partial y} + \left(q + \frac{ix}{2}\right)w\right|_{y=0} = 0 \quad \text{and} \quad \lim_{\substack{x\to 0\\ y>0}} \frac{w-2}{\sqrt{x}} = 0.$$

It is difficult to estimate the error introduced by discarding the "small" terms when using this method. To do this the well-known Fresnel "reflection" formula must also be considered. The essential advantage of the parabolic equation method is its great simplicity as well as the possibility of solving more complex problems (for example, wave diffraction from bodies of arbitrary shape).

In this work the first case considered is that in which the earth is assumed to be planar. Then the spherical earth case is considered and the same formulas are obtained by using the parabolic equation method as had been previously obtained by approximately summing the series which yield the exact solution of the problem. The agreement between results obtained by these two methods provides a justification for the use of the parabolic equation method in problems of radiowave diffraction from good conductors. Fock used this method widely in later works on diffraction.

In the work "Propagation of the Direct Wave Around the Earth with Due Account for Diffraction and Refraction",[5] the problem is solved under the assumption that the surface of the earth is homogeneous as well as that the dielectric constant of the air is a function $\varepsilon_e(h)$ only of the height $h = r - a$ of points above the horizon. A vertical dipole performing harmonic oscillations defined by the factor $e^{-i\omega t}$ is placed on the surface of the earth at the point $r = a$, $\theta = 0$.

A rapidly varying factor is isolated from the Hertz function U and a new "slowly" varying function U_2 is introduced by means of the formula

$$U = \frac{e^{iks}U_2}{\varepsilon_0(h)r\sqrt{\sin\theta}} \qquad (15)$$

where $s = a\theta$ is the length of arc on the terrestrial sphere from the point where the dipole is to the point above the earth at which the observer is situated.

The author neglects quantities of order $\left(\frac{2}{ka}\right)^{2/3}$ in the equation obtained for U_2. After introduction of the nondimensional variables x and y by means of the formulas

$$h = \sqrt[3]{\frac{a^*}{2k^2}}\, y \,; \qquad s = \sqrt[3]{\frac{2a^{*2}}{k}}\, x, \qquad (16)$$

where $a^* = \dfrac{1}{\dfrac{1}{a} + \dfrac{\varepsilon_0'(0)}{2\varepsilon_0(0)}}$ is the equivalent radius of the earth's surface, and after introducing the new function w_1 by means of the formula

$$U_2 = \frac{\varepsilon_0(0)\sqrt{a}}{\sqrt[6]{\frac{2a^{*2}}{k}}}\, w_1 \qquad (17)$$

the problem is reduced to determining the function $w_1(x,y)$ from the equation

$$\frac{\partial^2 w_1}{\partial y^2} + i\frac{\partial w_1}{\partial x} + y(1+g)w_1 = 0 \quad (y > 0) \qquad (18)$$

under the conditions

(19) $$\frac{\partial w_1}{\partial dy} + qw_1\Big|_{y=0} = 0; \quad \lim_{\substack{x\to 0\\ y>0}} \left(w_1 - \frac{2}{\sqrt{x}}\, e^{i\frac{y^2}{x}}\right) = 0,$$

and the natural radiation condition for $h >> 1$. The quantities q and g, entering in the formulas reduced above, have the following values

(20) $$q = i\sqrt[3]{\frac{ka^*}{2}}\sqrt{\frac{\mathcal{E}_o(0)}{\eta}}\,; \quad g = \frac{a^*}{2\,\mathcal{E}_o(0)}\left[\frac{\mathcal{E}_o(h) - \mathcal{E}_o(0)}{h} - \mathcal{E}_o'(0)\right].$$

Investigation of the equation for w_1 shows that if $g = 0$ and if the radius a is replaced by the equivalent radius of the earth a^*, then the mathematical problem is reduced to exactly the same form as when the atmosphere is absent. In the general case, g can be considered as a function of the product βy, where $\beta = \frac{1}{h_o}\sqrt[3]{\frac{a^*}{2k^2}}$ is a small parameter. The solution of the problem is successfully represented by the contour integral:

(21) $$w_1 = \frac{e^{i\frac{3}{4}\pi}}{\sqrt{\pi}} \int_\Gamma e^{ixt} \frac{f(y,t)}{\left(\frac{\partial f}{\partial y} + qf\right)_{y=0}}\, dt,$$

where $f(y,t)$ is an entire transcendental function with a definite behavior at infinity and satisfying the equations

(22) $$\frac{d^2f}{dy^2} + \left[y - t + yg(\beta y)\right] f = 0;$$

The contour Γ is infinite and encloses the first quadrant of the t plane.

Investigation of the solution of the problem constructed shows that the laws of geometric optics are correct in the line-of-sight region far from the horizon. The following inequality is the condition for this

$$\frac{y^2}{4x} = \frac{kh^2}{2s} >> 1 . \qquad (23)$$

The solution transforms into the Weyl-van der Pol formula for small values of x and y and for large values of $p = \sqrt[3]{\frac{ka^*}{2}} \cos \gamma$. The investigation of the solution in the half-shadow region permits the conclusion that the wave reaches the horizon with an amplitude and phase corresponding to the laws of geometric optics for an unbounded medium and undergoes diffraction according to the law of the focal field in the half-shadow region at the horizon.

This result agrees completely with the ideas of L.I. Mandel'shtam that the properties of the soil are essential not along the whole ray trajectory in radiowave propagation along the earth's surface but only in that region where the transmitter or receivers are located.

Let us turn to the work in which the problem diffraction from an arbitrary convex surface is analyzed.

An electromagnetic wave incident on a conductor excites surface currents which, in turn, are sources of scattered waves. Consequently, an essential step in the solution of the plane wave diffraction problem from a conductor of arbitrary shape is to find the currents excited on its surface.

In the work, "The Distribution of Currents Induced by a Plane Wave on the Surface of a Conductor",[6] the current distribution excited by a plane wave on the surface of a convex, perfectly conducting, sufficiently smooth body of arbitrary shape is analyzed under the condition that the length of the electromagnetic wave is very small in comparison with the body dimensions and the radii of curvature of its surface. A fundamental result of the work is the proof that the field has local character near the geometric shadow boundaries.

It is shown in the work that when the incident wave is polarized with the electric vector in the plane of incidence the current distribution near the boundaries cited is expressed through a universal (identical for all bodies) function $G(\xi)$ of the argument $\xi = \frac{\ell}{d}$, where ℓ is the distance from the geometric shadow boundaries measured in the incident plane and d is the width of the half-shadow region. An analytic expression is derived for the function $G(\xi)$ and detailed tables are given.

The solution of the problem of the current distribution is based essentially on the study of the solution of the integral equation for the current density $\vec{j}$ on the surface of the perfect conductor. If the monochromatic electromagnetic wave $\vec{H} = \vec{H}^{ex}\, e^{-ikct}$ falls on the conductor and if the following notation is introduced

(24) $$f = (1 - ikR)e^{ikR}; \quad \vec{j} = \frac{c}{4\pi}\left[\vec{n} \times \vec{H}^{ex}\right],$$

then the following integral equation is obtained for the surface current density

(25) $$\vec{j} = 2\vec{j}^{ex} + \frac{1}{2\pi}\int \frac{\vec{n} \times \left[\vec{j}' \times (\vec{r} - \vec{r}')\right]}{R^3} f \, ds',$$

where $\vec{n}$ is the unit vector normal to the conductor surface; $\vec{r}$ and $\vec{r}'$ are radius vectors of fixed points of the surface and of points with the surface element dS' and $R = |\vec{r} - \vec{r}'|$.

As an investigation of the integral equation in the case of very large values of k (i.e., small wavelengths λ) shows, it can be considered, with enough accuracy, that $\vec{j} = 2\vec{j}^{ex}$ on the illuminated part of the surface (which corresponds to Fresnel reflection theory and $\vec{j} = 0$ in the shadow part. In the neighborhood of the geometrical shadow boundaries, the integral equation shows that in a bandwidth of order

(26) $$d = \sqrt[3]{\frac{\lambda}{\pi} R_o^2},$$

where R_o is the radius of curvature of a section of the body surface by the incident plane, the current density and, therefore, the field has an approximate value dependent only on the value of the external field $\vec{H}^{ex}$ in the point under investigation, the geometric characteristics of the surface element and on the electric properties of the conductor. Such a result means that universal formulas for the current density on the surface of a perfect conductor in the half-shadow region can be obtained from the solution of the diffraction problem for the particular case of a convex surface. The universal formulas mentioned are obtained by considering the problem of plane wave diffraction from a paraboloid of revolution.

The result is

$$\vec{j} = \vec{j}^{ex} G(\xi) = \vec{j}^{ex} \frac{e^{i\frac{\xi^3}{3}}}{\sqrt{\pi}} \int_{\Gamma} \frac{e^{i\xi t}}{w'(t)} \, dt \,, \tag{27}$$

where $w(t)$ is the complex Airey function and Γ is a contour in the complex plane going from infinity to zero along the line $\arg = \frac{2}{3}\pi$ and from zero to infinity along the positive part of the real axis. An investigation of the asymptotic values of $G(\xi)$ for large positive and negative values of ξ shows that the current density $\vec{j}$ transforms continuously when the transition is made from the half-shadow into the line-of-sight or into the shadow regions, into the values $2\vec{j}^{ex}$ and $\vec{j} = 0$, respectively. Detailed tables are constructed for the function $G(\xi)$.

The result of the preceding work is generalized in "Field of a Plane Wave Near the Surface of a Conducting Body" in that, first, the field is determined not only on the body surface itself but also in a certain surface layer with thickness small in comparison with the radii of curvature; second, the body is considered to be not a perfect, but only a good conductor in the sense that the M.A. Leontovich conditions hold for the tangential field components on its surface. Furthermore, the polarization of the incident wave may be such that the electric vector lies in or is perpendicular to the plane of incidence.

Let us discuss the Fock work, "Fresnel Diffraction from Convex Bodies", (1951)[7].

Considered in this work is the diffraction from a sphere, wherein refraction of the atmosphere is not taken into account. It is considered that the source and the observer are above the surface of the earth, where h_1 is the source height and h_2 is the height of the observation point. The field is expressed through the two solutions U and w of the equation $\Delta U + k^2U = 0$. The following notations are introduced in addition to those used previously:

$$(28) \qquad y_1 = \left(\frac{ka}{2}\right)^{-\frac{1}{3}} kh_1; \quad y_2 = \left(\frac{ka}{2}\right)^{-\frac{1}{3}} kh_2 \;;$$

$$(29) \qquad q = \left(\frac{ka}{2}\right)^{\frac{1}{3}} (\eta + 1)^{-\frac{1}{2}} i; \quad q_1 = \left(\frac{ka}{2}\right)^{\frac{1}{3}} (\eta - 1)^{\frac{1}{2}} i \,.$$

The following formulas hold near the surface of the sphere:

$$(30) \qquad U = \frac{e^{ika\theta}}{a\sqrt{\theta \sin\theta}} V(x, y_1, y_2, q) \;;$$

$$(31) \qquad w = \frac{e^{ika\theta}}{a\sqrt{\theta \sin\theta}} V(x, y_1, y_2, q_1) \,,$$

and the attenuation factor V is expressed by a certain contour integral containing two Airey functions. All these results are contained in the work "Field from a Vertical and Horizontal Dipole, Raised Slightly Above the Earth's Surface", (1949)[8] and in the 1951 work, an approximate expression is given for V in the region of the shadow cone. Hence, it is considered that the parameter defined by the formula

$$\mu^2 = \frac{\sqrt{y_1 y_2}}{\sqrt{y_1} + \sqrt{y_2}}, \tag{32}$$

is large and the quantity $\xi = x - \sqrt{y_1} - \sqrt{y_2}$ is finite or small. Two functions are introduced

$$f(\alpha) = e^{-i\alpha^2 - i\frac{\pi}{4}} \frac{1}{\sqrt{\pi}} \int_{\alpha}^{\infty} e^{i\alpha^2} d\alpha \; ; \tag{33}$$

$$g(\alpha) = \frac{e^{i\frac{\pi}{4}}}{2\sqrt{\pi}} + \frac{\sqrt{-\alpha}\left(q + i\frac{\alpha}{2}\right)}{2\left(q - i\frac{\alpha}{2}\right)} e^{-\frac{i}{12}\alpha^2} . \tag{34}$$

Then the approximate expression $V(x_1 y_1, y_2, q)$ is the following for $\xi \geqslant 0$ in the shadow cone

$$V = \frac{\sqrt{x}}{\sqrt[4]{y_1 y_2}} e^{i\omega_0} \left[\mu f(\mu\xi) - g(\xi) + \frac{i}{4\mu^2} g''(\xi) \right] . \tag{35}$$

We do not cite the expression for ω_0. The principal term is $\mu f(\xi)$, proportional to the Fresnel integral. It is independent of the material of the diffracting body. Superimposed on the diffraction picture (Fresnel diffraction) determined by this term is the background dependent on the function $g(\xi)$ varies slowly in comparison with the principal term. This background depends on the material of the diffracting body.

REFERENCES

1. A translation of V.A. Fock, "Diffraction of Radiowaves around the Earth's Surface", is available through Morris D. Friedman, Inc.; Newtonville, Massachusetts.
2. See Appendix to this collection of Fock papers.
3. See Chapter VII of this collection.
4. See Chapter IV of this collection.
5. See Chapter VI of this collection.
6. See Chapter II of this collection.
7. See Chapter IX of this collection.
8. See Chapter VIII of this collection.

I. NEW METHODS IN DIFFRACTION THEORY

V. A. Fock

The general problem of the theory of diffraction of electromagnetic waves consists in finding a solution of Maxwell's equations, having prescribed singularities (field sources) and satisfying prescribed boundary conditions and conditions at infinity.

The solution of this problem presents serious mathematical difficulties, which arise chiefly from the necessity of taking into account the geometrical shape of the obstacles on which the wave is falling. The problem is somewhat simplified if only monochromatic waves of given frequency are considered, but the difficulties are still so great, that the problem has not yet been solved, except in cases when the obstacle is of a particularly simple form. The best known of these are the cases of a perfectly reflecting half-plane or a wedge, the cases of a sphere and a circular cylinder.

The cases of an elliptic and a parabolic cylinder have also been considered, and the field of a plane wave incident on a perfectly reflecting paraboloid of revolution (oblique incidence) has recently been obtained by the author. In the few cases enumerated a rigorous solution of the problem in the form of an infinite series of integrals has been obtained.

The aim of a theory is to give a picture reproducing all the qualitative and quantitative features of the phenomenon considered. This aim is not attained until the solution obtained is of a sufficiently simple form. If the rigorous solution has a complicated analytical form, it constitutes only the first step; a second step must be made - the derivation of formulas suitable for numerical calculations.

This second step may be as difficult as the first one. To give an example, we may mention that the problem of diffraction of electro-magnetic waves around a sphere was solved rigorously some 40 years ago (Mie). This problem includes that of the propagation of radio-waves along the surface of the earth. Owing to the slow convergence of the series involved, the general solution could, however, not be applied to the latter problem until 1918, when a transformation of the original series into another rapidly converging series was found (Watson). But the improved form of the solution was still unsatisfactory in some respects, being very complicated and applicable only in the region of the geometrical shadow (far beyond the line of horizon). A far more satisfactory form of the solution, applicable in all cases of practical importance, has been recently found by the author.[1] Thus, the way from the rigorous theoretical solution to the approximate practical one took about 40 years of research.

To find first a rigorous solution of a diffraction problem and then to transform it into another form suitable for numerical calculations - this straightforward method is, however, of a very limited application. It can only be applied to the few problems admitting a rigorous solution in form of series of integrals.

In other cases (especially when the diffracting obstacle is of arbitrary shape) attempts have been made to reduce the problem to integral equations. These attempts have proved successful from the theoretical point of view; but with the exception of a paper by the author,[2] no use has been made of the integral equations for the practical solution of the problem, the general theory of integral equations being quite useless for purposes of numerical calculation.

An approximate method, sufficiently general and leading to sufficiently simple formulas is thus urgently needed. In the following we shall outline the principal ideas of such a method, proposed and developed by the author.

Every approximate method is based on the smallness of some parameters involved in the problem. We have to consider which of the parameters of our problem may be regarded as small.

We are usually concerned with the propagation of waves in air, i.e., in a medium with properties widely different from those of the scattering bodies (obstacles). The electrical properties of these bodies are characterized by means of the complex dielectric permeability

$$\eta = \epsilon + i\,\frac{4\pi\tau}{\omega} \tag{1}$$

(ϵ denotes as usual the dielectric constant, τ - the conductivity of the medium, ω - the frequency). Now it is essential that in most cases $|\eta| \gg 1$. Thus we may choose as one of the small parameters of the problem the inverse value of $|\eta|$ or the quantity $1:\sqrt{|\eta|}$.

Next, the wave-length λ in vacuo is usually very much smaller than the radii of curvature of the scattering bodies. We thus have another small parameter - the quotient $\lambda:R$, where R is the radius of curvature of the obstacle. It is convenient to take instead the quantity

$$\frac{1}{m} = \sqrt[3]{\frac{\lambda}{\pi R}} \tag{2}$$

In addition to the two small parameters defined above, there may be others, depending on the position of the point of observation. For instance, in the problem of the propagation of radio waves along the earth surface the angle of inclination of the ray to the horizon may be regarded as small.

Let us consider the consequences of the fact that the parameters $1:\sqrt{|\eta|}$ and $1:m$ are small. In the limiting case $|\eta| = \infty$ (perfect conductor) a great simplification arises from the fact that the field is known beforehand inside the conductor (this field being equal to zero). We can confine ourselves to the space outside the conductor by prescribing proper boundary conditions to the field in air (the tangential components of the electrical vector should vanish at the surface). A similar situation arises if $\sqrt{|\eta|}$ is very large. The field inside the body is in this case very small except in a thin surface layer (skin-effect), and the influence of this layer may be accounted for by stating boundary conditions for the external field. These are of the form

$$\frac{4\pi}{c} j_x = \sqrt{\eta}(E_x - n_x E_n) = n_y H_z - n_z H_y \text{ , etc.,} \qquad (3)$$

where (j_x, j_y, j_z) is the surface current density vector, (n_x, n_y, n_z) the unit vector of the normal to the surface, E_n the normal component of the electric field, the meaning of the other symbols being evident. These conditions, first stated by Leontovich[3] in a somewhat different form, apply if $|\eta| \gg 1$ and if $KR\sqrt{|\eta|} \gg 1$ $(K = 2\pi : \lambda)$. The latter inequality signifies that the thickness of the skin layer should be small as compared with the radius of curvature of the obstacle. Conditions (3) may be easily generalized for arbitrary values of the magnetic permeability m.

Consequently the smallness of $1 : \sqrt{|\eta|}$ permits us to confine our attention to the field outside and on the body, which constitutes an important simplification of the problem.

We now proceed to examine the influence of the smallness of the wave-length.

As well known, in the limiting case of small wave-lengths the laws of geometrical optics become valid. Particularly, the boundary of the shadow on the surface of the body becomes sharp and well defined. On the one side of the boundary — in the illuminated region — the field obeys very nearly Frensnel's laws of reflection, and on the dark side the field rapidly decreases to zero.

The approximation given by the geometrical optics is, however, not sufficient for our purposes. The point of interest for us is the diffraction phenomenon in its strict sense, i.e., the bending of the ray around the obstacle. This phenomenon cannot be treated by the means of geometrical optics, and to give a theory of this phenomenon a more accurate solution of the field equations is required.

The author succeeded in finding this solution by means of a new principle which may be called "The Principle of the Local Field in the Penumbra Region".

This principle consists in the following: - The transition from light to shadow on the surface of the body takes place in a narrow strip along the boundary of the geometrical shadow. The width of this strip is of the order

$$d = \sqrt[3]{\frac{\lambda}{\pi} R_0^2}\ , \qquad (4)$$

where R_0 is the radius of curvature of the normal section of the body by the plane of incidence. It may be proved that, with neglect of small quantities of the order $\sqrt[3]{\frac{\lambda}{\pi R_0}}$, the field in this strip has a local character: it depends only on the value of the field of the incident wave in the neighborhood of the point considered, on the geometrical shape of the body near the point and on the electrical properties of the material of the body. The field near a given point on the strip does not depend on its values at distant points and can be calculated separately.

To establish the principle of the local field and to derive explicit formulas for this field we have used two different methods.

One of these (2) applies to the case of an absolute conductor and gives the values of the field on its surface. We start with the integral equation for the surface current density j. This is of the form

$$j = 2j^{ex} + \frac{1}{2\pi}\int \left\{ \frac{nx\left[j'x(z - z')\right]}{R^3} f \right\} \frac{dS'}{s\omega if} \tag{5}$$

where

$$f = (1 - iKR)e^{iKR} \tag{6}$$

The vector j^{ex} (external current density) is defined by the expression (3), where H is replaced by H^{ex}, the magnetic vector of the external field; z is the radius vector of the point of observation, z' that of the point of integration; $R = \left| z - z' \right|$ is the length of the chord between z and z';

n is the value of the unit vector of the normal at z. A qualitative study of the integral equation permits us to establish the principle of the local field. This principle once established, we have to find a solution of the diffraction problem for a convex body of a particular shape and to derive approximate formulas for the field on its surface. In virtue of the principle of the local field, these formulas hold for any other convex body having at the point considered the same values of the principal radii of curvature. (The particular body must of course be sufficiently general to possess points with any prescribed values of principal radii of curvature; actually a paraboloid of revolution has been used). Proceeding in this way we arrive at a general formula for the surface values of the tangential components of the magnetic field or, which amounts to the same, for the surface current density vector. This formula is of the form

$$j = j^{ex} G(\epsilon, 0) \tag{7}$$

where the argument ϵ in G denotes the quantity

$$\epsilon = \ell : \sqrt[3]{\frac{\lambda}{\pi} R_o^2} = \ell : d , \tag{8}$$

ℓ being the distance from the boundary of the geometrical shadow, measured along the ray (i.e., along the line of intersection of the plane of incidence with the surface of the body) and taken positive in the direction of the shadow and negative in the opposite direction. The function $G(\epsilon, 0)$ is defined by the integral

$$G(\epsilon, 0) = \left(e^{\frac{i\epsilon^3}{3}}\right) \frac{1}{\sqrt{\pi}} \int_C \frac{e^{i\epsilon t}\,dt}{\omega'(t)} , \qquad (9)$$

where C is a contour in the complex t-plane running from infinity to zero along the line arc $t = \frac{2\pi}{3}$ and from zero to infinity along the positive real axis.

The function $\omega(t)$ may be called the complex Airy's function; it is defined by the differential equation

$$\omega''(t) = t\omega(t) \qquad (10)$$

and by the asymptotic behavior for large negative values of t

$$\omega(t) = e^{i\frac{\pi}{4}} (-t)^{-1/4} \cdot \exp\left[i\, \frac{2}{3}\, (-t)^{3/2}\right]. \qquad (11)$$

The function $G(\epsilon,0)$ tends to the limit $G = 2$ for large negative values of ϵ, while its modulus decreases exponentially for large positive values of ϵ. Formula (7) reproduces thus the gradual decrease of the field amplitude when passing from light to shadow.

The same results may be obtained by another method[4] which allows us to generalize them in two respects. Firstly, the body need not be a perfect conductor, but may have a finite conductivity, if only the boundary conditions (3) are applicable. Secondly, the field is obtained not only on the surface of the body, but also near the surface (at distances that are small as compared with the radii of curvature). The method consists in simplifying Maxwell's equations and boundary conditions by neglecting quantities of the order of the square of the small parameters $1-\sqrt{|\eta|}$ and $1 : m$. The wave equation for the amplitude is thereby replaced by a parabolic equation of Schrodinger's type. The simplified equations are valid in a limited region near a point on the penumbra strip.

The solution of these equations may be performed by means of the separation of variables and yields the field in the region considered and especially in the penumbra strip on the body. Introducing the complex quantity

$$q = \frac{im}{\sqrt{\eta}} = \frac{i}{\sqrt{\eta}} \cdot \sqrt[3]{\frac{\pi R_o}{\lambda}} \qquad (12)$$

(the modulus $|q|$ is thus the quotient of the two small parameters), we may write instead of (7)

$$J = J^{ex}\, G(\epsilon,\ q)\ , \qquad (13)$$

where

$$G(\epsilon,\ q) = e^{\frac{i\epsilon^3}{3}} \cdot \frac{1}{\sqrt{\pi}} \int_C \frac{e^{i\epsilon t}\, dt}{\omega'(t) - q\omega(t)}\ , \qquad (14)$$

the contour C being the same as in (9). These formulas give thus the distribution of currents on the penumbra strip on the body and generalize our previous formulas (7) and (9). The formulas for the field near the surface are more complicated and will not be written here.

It is to be noted that in the outward portion of the strip, where the illuminated region begins, approximate expressions can be derived from our formulas that coincide with expressions for the field obtained by superposing the incident and the reflected wave and using Fresnel's coefficients of reflection. On the other hand, in the opposite portion of the strip the field is practically zero. Thus our formulas constitute the missing link joining the two regions where the laws of geometrical optics may be applied. Together with Fresnel's formulas they allow us to compute the field near and on the whole surface of the diffracting body.

In some problems this is all that is required. In the problem of propagation of waves around the earth's surface, for instance, we are only concerned with the field on heights not exceeding ten kilometers--a quantity that is small as compared with the earth's radius (6380km.). In this instance our formulas, if modified so as to include the case when the source is near or on the surface, give the required solution.

In other problems, however, the field at large distances from the scattering body is needed. In spite of the fact that our formulas are valid only in the region near the surface, they provide a means to calculate the field at large distances also. Indeed, the field of the scattered wave is generated by the currents induced on the surface (in the skin-layer) by the incident wave. These currents are given by our formulas. Thus, by applying well-known theorems on the vector potential due to a given current distribution, we may, in principle, calculate the field for arbitrary distances from the reflecting body.

The principle of the local field in the penumbra region provides thus a basis for the approximate solution of the problem of diffraction in the general case of a convex body of arbitrary shape.

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II. THE DISTRIBUTION OF CURRENTS INDUCED BY A PLANE WAVE ON THE SURFACE OF A CONDUCTOR

V. Fock

The distribution of currents, induced on the surface of an perfectly conducting body by an incident plane wave is considered. The body is supposed to be convex and to have a continuously varying curvature. The wave length λ of the incident wave is supposed to be small as compared with the dimensions of the body and with the radii of curvature of its surface. It is shown that the current distribution in the vicinity of the geometrical shadow is expressible in terms of an universal function $G(\xi)$ (the same for all bodies), depending on the argument $\xi = \ell/d$, where ℓ is the distance from the boundary of the geometrical shadow, measured in the plane of incidence, and d is the width of the penumbra region $\left(d = \sqrt[3]{\frac{\lambda}{\pi} R_o^2},\ R_o\right.$ is the radius of curvature of the normal section of the body by the plane of incidence$\left.\right)$. For the function $G(\xi)$ an analytical expression is derived and tables are computed.

Let us consider a perfectly conducting body on the surface of which a plane electromagnetic wave is incident. The surface of the conductor is supposed to be convex, with a continuously varying curvature. The incident wave induces on the conductor electrical currents, which in their turn become a source of the scattered wave. If the current distribution on the conductor is determined, then the calculation of the field of the scattered wave may be performed by applying the well-known formulas for the vector-potential. Hence the essential step in solving the problem of diffraction of a plane wave by a perfect conductor is to find the currents induced on its surface.

The present paper is a preliminary report on our work concerning the approximate solution of this problem.

1. Let us denote by j the surface current density on the conductor. The vector j is defined for every point on the surface and is directed along the tangent to the surface. It is completely determined by its two tangential components, the third component (normal to the surface) being equal to zero.

It may be shown that the vector j satisfies the following integral equation:

$$j = 2j^{ex} + \frac{1}{2\pi}\int\left\{\frac{n \times [j' \times (r - r')]}{R^3} f\right\}_{surf} dS' \qquad (1.01)$$

with

$$f = (1 - ikR)e^{ikR} \quad . \qquad (1.02)$$

In this equation R is the length of the chord joining the two points of the surface: the fixed point $r(x,y,z)$, for which the integral is evaluated, and the variable point $r'(x',y',z')$, whose coordinates are functions of the integration variables. n is a unit vector of the normal to the surface at the point r, dS' is the surface element at r' and k is the absolute value of the wave vector.

The quantity j^{ex} is an "external" current density defined by the formula

$$j^{ex} = \frac{c}{4\pi}\left[n \times H^{ex}\right], \qquad (1.03)$$

where H^{ex} is the value of the magnetic field of the incident wave on the surface ("external" field).

If the dependence of the external field upon the coordinates is given by the factor

$$e^{ik(\alpha x+\beta y+\gamma z)}\ , \tag{1.04}$$

then the current density may be sought in the form of a product of a similar factor with a slowly varying function of coordinates. The integral (1.01) after dividing by (1.04) takes the form

$$I = \int e^{ik\left[R+\alpha(x'-x)+\beta(y'-y)+\gamma(z'-z)\right]}\ \phi dS'\ , \tag{1.05}$$

where ϕ is a slowly varying function. If the wave length is sufficiently small as compared with the dimensions of the body, the value of the integral will be approximately

$$I = \frac{2\pi i}{k}\ \frac{R}{\cos\theta}\ \phi\ , \tag{1.06}$$

where the point x' y' z' is connected with the point x y z as it is shown in Figs. 1 and 2, and θ is the angle of incidence of the ray.

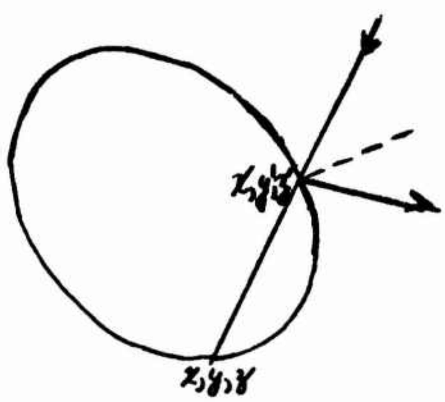

Fig. 1

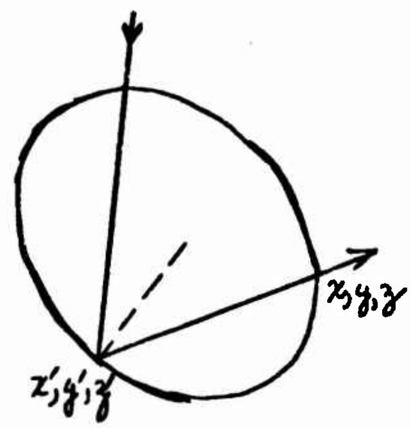

Fig. 2

The analytical connection between the points x' y' z' and x y z is given by the following formulas. Let n' denote the unit vector of the normal at the point x' y' z' and let

$$\left.\begin{aligned} \alpha + 2n'_x \cos\theta &= \alpha' , \\ \beta + 2n'_y \cos\theta &= \beta' , \\ \gamma + 2n'_z \cos\theta &= \gamma' , \end{aligned}\right\} \qquad (1.07)$$

where

$$\cos\theta = -(\alpha n'_x + \beta n'_y + \gamma n'_z) . \qquad (1.08)$$

The quantities α', β', γ' are the direction cosines of the ray reflected at the point x' y' z'.

With these notations, we have either:

$$\frac{x - x'}{R} = \alpha \; ; \quad \frac{y - y'}{R} = \beta \; ; \quad \frac{z - z'}{R} = \gamma \qquad (1.09)$$

or

$$\frac{x - x'}{R} = \alpha' \; ; \quad \frac{y - y'}{R} = \beta' \; ; \quad \frac{z - z'}{R} = \gamma' , \qquad (1.10)$$

the formulas (1.09) being valid, if the point x' y' z' is situated on the illuminated part of the surface (Fig. 1), while (1.10) are valid, if this point is situated on the shadow part of the surface. In the latter case the "reflected" ray is fictitious.

With the same degree of approximation as in formula (1.06) the integral equation (1,01) allows the following solution:

$$\left.\begin{aligned} j &= 2j^{ex} \text{ on the illuminated part,} \\ j &= 0 \text{ on the shadow part.} \end{aligned}\right\} \qquad (1.11)$$

Near the boundary of the geometrical shadow (where $\cos\theta \approx 0$), formula (1.06) ceases to be valid and expression (1.11) does not give a gradual transition from light to shadow.

2. In order to obtain for the currents an expression valid in the transition region also, it is necessary to use a more exact solution. It is rather difficult to derive it directly from the integral equation, but we have succeeded to obtain it in an indirect way, on the basis of the following considerations.

First of all, it is seen from Figs. 1 and 2 that if the point x y z lies near the geometrical boundary of the shadow, the point x' y' z' lies also near this boundary and near the point x y z. Therefore, the value of the integral (1.01) is determined by the values of the integrand in the neighborhood of the point for which the integral is evaluated. Thus, in the region of the penumbra (near the geometrical boundary of the shadow) the field has a local character. Secondly, the investigation of the integral equation (carried out under the assumption that the chord can be replaced by its projection on the tangent plane) shows that the width of the penumbra region is of the order of

$$d = \sqrt[3]{\frac{\lambda}{\pi} R_o^2} , \qquad (2.01)$$

where R_o is the radius of curvature of the section of the body surface by the plane of incidence. But in a region of width d and in a certain more extended region the nucleus of the integral equation depends essentially only on the curvature of the surface in the neighborhood of a given point (i.e. on the second but not on the higher derivatives of the surface equation with respect to coordinates).

Hence it follows, that all bodies with a smoothly varying curvature have the same current distribution in the penumbra region, if only the curvatures and the incident wave are the same near the point under consideration.

The results stated permit us to infer that, if we solve the problem for any particular case, we can obtain universal formulas for the field on the surface of a perfect conductor. These formulas immediately apply to the region of the penumbra, but the field may be considered as known everywhere on the surface, since for the illuminated region and for the remote shaded region the expressions (1.11) are valid.

3. The derivation of these universal formulas is too complicated to be given in any detailed form in a short paper. We confine ourselves to some indications as to the method, and to the statement of the result, which may be done in quite a simple way.

The considerations developed above show, that for the derivation of the general formulas we can start from an exact solution of the problem of diffraction of a plane wave by some convex body with a smoothly varying curvature. The surface of the body must, of course, be sufficiently general, i.e. must possess points with given values of the principal radii of curvature.

There are two cases in which exact solutions of the problem are known, namely, the case of a sphere and the case of a circular cylinder (in the last case the incidence of the wave is supposed to be normal). These bodies are, however, not sufficiently general: for a sphere the two radii of curvature are equal, and for a cylinder one of the radii is infinite. The simplest of the bodies having arbitrary values of the curvature radii are: the ellipsoid and the paraboloid of revolution. For these bodies only the general form of the solution of the scalar wave equation is known; the complete solution of Maxwell's equation for the given physical problem appears to be unknown.

In our work we have obtained the required solution for the paraboloid of revolution (particularly the values of the tangential components of the magnetic field on its surface) and have used this solution to derive the approximate formulas.

Let the equation of the paraboloid have the form

$$x^2 + y^2 - 2az - a^2 = 0 . \qquad (3.01)$$

The components of the field of the incident wave are

$$\left.\begin{aligned} E_x &= E_0 \cos\delta\, e^{i\Omega} , & H_x &= 0, \\ E_y &= 0 , & H_y &= E_0\, e^{i\Omega} , \\ E_z &= - E_0 \sin\delta\, e^{i\Omega} ; & H_z &= 0 , \end{aligned}\right\} \qquad (3.02)$$

where

$$\Omega = k\,(x \sin\delta + z \cos\delta) . \qquad (3.03)$$

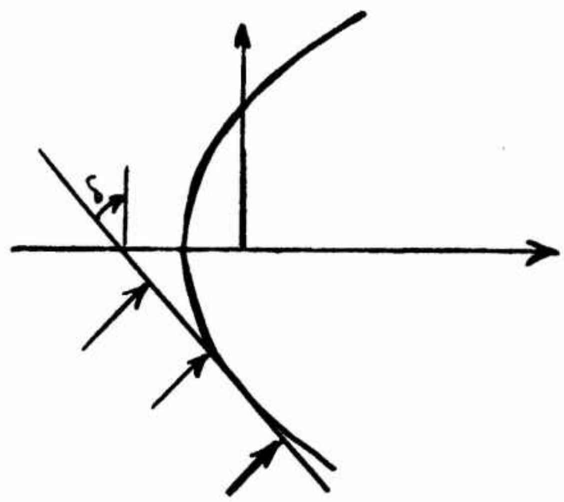

Fig. 3

If the parabolic coordinates:

$$\begin{aligned} u &= k\,(r + z) ; \\ v &= k\,(r - z) ; \\ \phi &= \operatorname{arc\,tg} \frac{y}{x} \end{aligned} \qquad (3.04)$$

with

$$r = \sqrt{x^2 + y^2 + z^2} \,, \tag{3.05}$$

are introduced, the equation of the paraboloid becomes

$$\upsilon = \upsilon_0 = ka \,. \tag{3.06}$$

For the generalized (covariant) tangential components of the external magnetic field we have the expressions:

$$2iu\, H_u^{ex} + H_\phi^{ex} = \frac{E_0}{k} \sqrt{u\upsilon}\; e^{i\Omega + i\phi} \,, \tag{3.07}$$

$$-\, 2iu\, H_u^{ex} + H_\phi^{ex} = \frac{E_0}{k} \sqrt{u\upsilon}\; e^{i\Omega - i\phi} \,. \tag{3.08}$$

In the new coordinates the expression for Ω has the form

$$\Omega = \frac{1}{2}\,(u - \upsilon) \cos\delta + \sqrt{u\upsilon}\, \sin\delta \cos\phi \,. \tag{3.09}$$

For the same components of the total field expressions in form of Fourier series with respect to the angle ϕ are obtained. The coefficients of $\sin s\phi$ and $\cos s\phi$ in these series are definite integrals with respect to the parameter t, involving some complicated functions of u, υ, δ, s, t. These series and integrals can be transformed into double integrals of the form

$$2iu\, H_u + H_\phi = \frac{E_0 \sqrt{u\upsilon}}{2\pi\, k \sin\delta} \int\!\!\int g(s,t)\, e^{-is\phi + it\, \mathrm{lgtg}\, \frac{\delta}{2}}\, ds\, dt \,, \tag{3.10}$$

where the function $g(s,t)$ is defined in the following way. Let $\zeta\,(\upsilon,s,t)$ be an integral of the differential equation

$$\upsilon \frac{d^2\zeta}{d\upsilon^2} + \frac{d\zeta}{d\upsilon} + \left(\frac{\upsilon}{4} - \frac{s^2}{4\upsilon} + \frac{t}{2}\right) \zeta = 0 \qquad (3.11)$$

having at $\upsilon \to \infty$ an asymptotic expression

$$\zeta(\upsilon,s,t) = e^{-\frac{\pi}{4} t - i \frac{s+1}{4} \pi} \upsilon^{-\frac{1}{2} + \frac{it}{2}} e^{i \frac{\upsilon}{2}} F_{20} \left(\frac{1 - s - it}{2} ; \frac{1 + s - it}{2} ; - \frac{i}{\upsilon}\right) \qquad (3.12)$$

where F_{20} is an asymptotic series of the form

$$F_{20} \left(\alpha, \beta, \frac{1}{x}\right) = 1 + \frac{\alpha\beta}{1} \frac{1}{x} + \frac{\alpha(\alpha+1)\ \beta(\beta+1)}{1 \cdot 2} \frac{1}{x^2} + \ldots \qquad (3.13)$$

We put

$$N(s,t) = \frac{1}{\pi} e^{\pi t - i \frac{\pi}{2} s} \frac{\Gamma\left(\frac{s-it}{2}\right) \Gamma\left(\frac{-s-it}{2}\right)}{\zeta^2 (\upsilon,s,-t-i) + \frac{1}{4} (s^2+t^2) \zeta^2 (\upsilon,s,-t+i)}, \qquad (3.14)$$

where υ is considered to be the quantity (3.06).

Then

$$g(s,t) = e^{is \frac{\pi}{2}} \zeta (u,s+1,t) \zeta (\upsilon,s-1,t) (s-it) N (s,t) . \qquad (3.15)$$

With $g(s,t)$ having this value, the expression (3.10) is valid, if $-\pi/2 < \phi < \pi/2$. In the cases $\pi/2 < \phi < 3\pi/2$ and $-3\pi/2 < \phi < -\pi/2$ we have to take for $g(s,t)$ a somewhat different expression, which we shall not write down here. The integration in (3.10) with respect to the variable t is to be made along the real axis from $-\infty$ to $+\infty$ and with respect to s along the imaginary axis from $-i\infty$ to $+i\infty$. The value of $-2iu\,H_u + H_\phi$ is obtained from (3.10), if we replace ϕ by $-\phi$.

The double integral can be evaluated approximately under the assumption, that the value of $\upsilon = ka$ is very large. Let us introduce the quantity

$$\xi = \frac{\sqrt{u\upsilon}\,\sin\delta\cos\phi - \upsilon\cos\delta}{\left[2\upsilon\,(u+\upsilon)\right]^{1/3}(\sin\delta)^{2/3}} . \tag{3.16}$$

It is easy to verify that on the geometrical boundary of the shadow $\xi = 0$; but in general ξ will be large, of the order of $\upsilon^{1/3}$. Therefore, when evaluating the integrals we shall consider υ to be very large and ξ to be arbitrary (in general, finite). It can be shown, that under these assumptions the following approximate expressions for the integrals are valid with a relative error of the order of $\upsilon^{-1/3}$:

$$2iu\,H_u + H_\phi = \frac{E_o}{k}\sqrt{u\upsilon}\,e^{i\Omega + i\phi}\,G(\xi) , \tag{3.17}$$

$$-2iu\,H_u + H_\phi = \frac{E_o}{k}\sqrt{u\upsilon}\,e^{i\Omega - i\phi}\,G(\xi) , \tag{3.18}$$

where

$$G(\xi) = e^{i\frac{\xi^3}{3}}\frac{1}{\sqrt{\pi}}\int_{\Gamma_1}\frac{e^{i\xi\tau}\,d\tau}{w'(\tau)} \tag{3.19}$$

the symbol Γ_1 denoting a contour running from infinity to the origin along the ray arc $z = 2/3\,\pi$ and from the origin to infinity along the ray arc $z = 0$ (the positive real axis).

The function $w(\tau)$ whose derivative is involved in the integrand has been studied in our previous paper*. $w(\tau)$ satisfies the differential equation

$$w''(\tau) = \tau w(\tau) , \qquad (3.20)$$

and can be written in the form of an integral

$$w(\tau) = \frac{1}{\sqrt{\pi}} \int_{\Gamma_2} e^{\tau z - \frac{1}{3} z^3} dz , \qquad (3.21)$$

where the contour denoted by Γ_2 runs from infinity to the origin along the arc $z = -2/3\ \pi$ and from the origin to infinity along the positive real axis.

Comparison of (3.17) and (3.18) with (3.07) and (3.08) gives

$$H_{tg} = H_{tg}^{ex} G(\xi) . \qquad (3.22)$$

Thus the tangential components of the total magnetic field are equal to the tangential components of the external field multiplied by a certain complex function of a single variable ξ. A similar relation exists between the total and the "external" current density, namely

$$j = j^{ex} G(\xi) . \qquad (3.23)$$

Let us examine the geometrical meaning of the variable ξ in more detail. Consider the section of the paraboloid surface by the plane of incidence passing through the given point (Fig. 4). We denote by ℓ the distance of the given point from

*Journ. of Phys., 9:255, 1945.

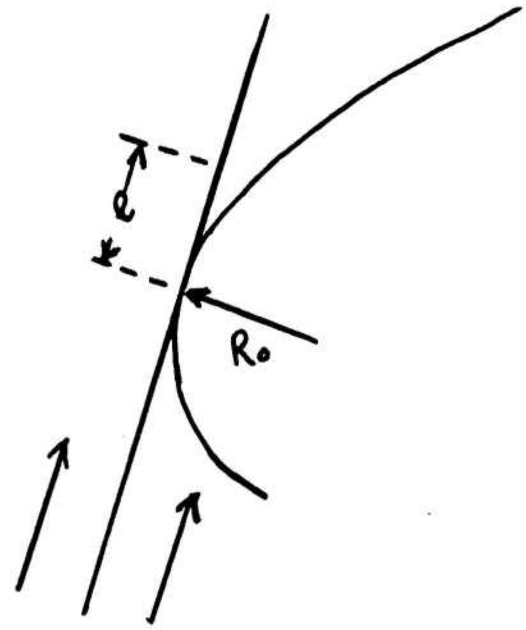

Fig. 4

the geometrical boundary of the shadow, considered positive in the direction of the shadow and negative in the direction of the light. The distance ℓ is measured in the plane of incidence. Let R_o be the radius of curvature of the surface section and $k = 2\pi/\lambda$ the absolute value of the wave vector.

Then the quantity

$$\xi = \sqrt[3]{\frac{k}{2R_o^2}}\ \ell = \frac{\ell}{d} \qquad (3.24)$$

[where d is the width (2.01) of the penumbra region] is easily seen to coincide with the quantity (3.16) defined for a paraboloid of revolution. Since we know beforehand that formulas (3.22) and (3.23) are quite general, we conclude that they are valid for all bodies with a given curvature, if ξ is given by (3.24).

These formulas give the transition from the shadow to the light.

For large positive values of ξ the function $G(\xi)$ is approximately equal to

$$G(\xi) = ce^{i\left(\frac{\xi^3}{3} + a\xi\right)} e^{-b\xi} , \qquad (3.25)$$

where a, b, c are known numbers; namely

$$a = 0.5094 \ ; \quad b = 0.8823 \ ; \quad c = 1.8325 \ . \qquad (3.26)$$

Owing to the factor $e^{-b\xi}$ the function $G(\xi)$ decreased rapidly. This corresponds to the decrease of the amplitude in the shadow region.

For large negative values of ξ the function $G(\xi)$ admits an asymptotic expansion of the form

$$G(\xi) = 2 + \frac{i}{2\xi^3} + \ . \ . \ . \qquad (3.27)$$

and tends to a limit which is equal to 2. This limiting value corresponds to formulas (1.11) for the illuminated region. The discontinuous function (1.11) is thus replaced in our more exact solution by the continuous function (3.23). This enables us to calculate the distribution of currents on the surface of a conducting body with sufficient accuracy.

In the Appendix are given tables of the function G defined by (3.19) and of the function g related to G by the equation

$$G(x) = e^{i\frac{x^3}{3}} g(x) \qquad (3.28)$$

and expressible in form of the integral

$$g(x) = \frac{1}{\sqrt{\pi}} \int_{\Gamma_1} \frac{e^{ixt}}{w'(t)}\, dt \,. \tag{3.29}$$

The function $G(x)$ is tabulated for values of x from $x - 4.5$ to $x = 1$ with interval 0.1, and the function $g(x)$ is tabulated for a range of values of x from $x = -1$ to $x = 4.5$ with the same interval. For values of x less than $x = -4.5$ expression (3.27) may be used, and for values of x greater than $x = 4.5$ formula (3.25) becomes applicable.

APPENDIX

Table of the function $G(x) = e^{i\frac{x^3}{3}}\, g(x)$

x	Re G	Im G	| G |	arc G
- 4.5	1.9998	-0.0055	1.9998	- 9'30"
- 4.4	1.9997	-0.0059	1.9997	- 10'10"
- 4.3	1.9997	-0.0063	1.9997	- 10'50"
- 4.2	1.9996	-0.0067	1.9997	- 11'40"
- 4.1	1.9996	-0.0073	1.9996	- 12'30"
- 4.0	1.9995	-0.0078	1.9995	- 13'20"
- 3.9	1.9994	-0.0084	1.9995	- 14'30"
- 3.8	1.9994	-0.0090	1.9994	- 15'30"
- 3.7	1.9992	-0.0098	1.9993	- 16'50"
- 3.6	1.9991	-0.0106	1.9991	- 18'10"
- 3.5	1.9990	-0.0115	1.9990	- 19'40"
- 3.4	1.999	-0.012	1.999	- 21'
- 3.3	1.999	-0.014	1.999	- 23'

x	Re G	Im G	|G|	arc G
- 3.2	1.998	-0.015	1.998	- 26'
- 3.1	1.998	-0.016	1.998	- 28'
- 3.0	1.998	-0.018	1.998	- 31'
- 2.9	1.997	-0.020	1.997	- 34'
- 2.8	1.996	-0.022	1.996	- 37'
- 2.7	1.996	-0.024	1.996	- 41'
- 2.6	1.995	-0.026	1.995	- 46'
- 2.5	1.993	-0.029	1.994	- 51'
- 2.4	1.992	-0.033	1.992	- 56'
- 2.3	1.990	-0.036	1.990	- 1°03'
- 2.2	1.988	-0.040	1.988	- 1°10'
- 2.1	1.985	-0.045	1.985	- 1°18'
- 2.0	1.981	-0.050	1.982	- 1°27'
- 1.9	1.977	-0.056	1.977	- 1°37'
- 1.8	1.971	-0.062	1.972	- 1°47'
- 1.7	1.965	-0.068	1.966	- 1°58'
- 1.6	1.956	-0.075	1.958	- 2°11'
- 1.5	1.946	-0.082	1.948	- 2°25'
- 1.4	1.933	-0.090	1.936	- 2°40'
- 1.3	1.919	-0.098	1.921	- 2°55'
- 1.2	1.901	-0.105	1.904	- 3°10'
- 1.1	1.880	-0.113	1.884	- 3°27'
- 1.0	1.857	-0.119	1.861	- 3°40'
- 0.9	1.829	-0.123	1.833	- 3°51'
- 0.8	1.798	-0.126	1.802	- 4°00'
- 0.7	1.762	-0.126	1.766	- 4°05'
- 0.6	1.722	-0.122	1.726	- 4°03'
- 0.5	1.678	-0.115	1.682	- 3°54'
- 0.4	1.630	-0.103	1.633	- 3°36'
- 0.3	1.578	-0.086	1.580	- 3°06'
- 0.2	1.522	-0.063	1.523	- 2°22'
- 0.1	1.462	-0.034	1.463	- 1°21'

x	Re G	Im G	|G|	arc G
0	1.399	0	1.399	0°00'
0.1	1.333	0.040	1.334	1°44'
0.2	1.263	0.086	1.266	3°55'
0.3	1.189	0.137	1.197	6°35'
0.4	1.111	0.193	1.128	9°51'
0.5	1.029	0.252	1.059	13°45'
0.6	0.941	0.312	0.991	18°22'
0.7	0.846	0.373	0.924	23°47'
0.8	0.744	0.432	0.860	30°08'
0.9	0.634	0.484	0.798	37°22'
1.0	0.515	0.529	0.738	45°44'

Table of the function $g(x) = e^{-i\frac{x^3}{3}}\, G(x)$

x	Re g	Im g	|g|	arc g
- 1.0	1.794	0.495	1.861	15°26'
- 0.9	1.805	0.320	1.833	10°04'
- 0.8	1.793	0.181	1.802	5°47'
- 0.7	1.765	0.076	1.766	2°28'
- 0.6	1.726	0.002	1.726	0°04'
- 0.5	1.681	- 0.045	1.682	- 1°31'
- 0.4	1.632	- 0.068	1.633	- 2°23'
- 0.3	1.578	- 0.071	1.580	- 2°35'
- 0.2	1.522	- 0.059	1.523	- 2°13'
- 0.1	1.462	- 0.034	1.463	- 1°20'
0	1.399	0	1.399	0°00'

x	Re g	Im g	|g|	arc g
0.1	1.333	0.040	1.334	$1^{\circ}43'$
0.2	1.263	0.083	1.266	$3^{\circ}45'$
0.3	1.190	0.127	1.197	$6^{\circ}04'$
0.4	1.115	0.169	1.128	$8^{\circ}37'$
0.5	1.038	0.209	1.059	$11^{\circ}21'$
0.6	0.961	0.244	0.991	$14^{\circ}14'$
0.7	0.883	0.274	0.924	$17^{\circ}14'$
0.8	0.806	0.299	0.860	$20^{\circ}19'$
0.9	0.732	0.317	0.798	$23^{\circ}27'$
1.0	0.660	0.331	0.738	$26^{\circ}38'$
1.1	0.591	0.339	0.682	$29^{\circ}50'$
1.2	0.527	0.343	0.628	$33^{\circ}02'$
1.3	0.467	0.342	0.578	$36^{\circ}13'$
1.4	0.411	0.338	0.532	$39^{\circ}25'$
1.5	0.360	0.330	0.488	$42^{\circ}34'$
1.6	0.313	0.320	0.448	$45^{\circ}42'$
1.7	0.270	0.309	0.410	$48^{\circ}48'$
1.8	0.232	2.960	0.376	$51^{\circ}53'$
1.9	0.197	0.281	0.343	$54^{\circ}56'$
2.0	0.167	0.267	0.315	$57^{\circ}59'$
2.1	0.140	0.252	0.289	$61^{\circ}00'$
2.2	0.116	0.237	0.264	$64^{\circ}00'$
2.3	0.095	0.222	0.242	$66^{\circ}58'$
2.4	0.076	0.208	0.221	$69^{\circ}56'$
2.5	0.0596	0.1936	0.2025	$72^{\circ}54'$
2.6	0.0453	0.1797	0.1853	$75^{\circ}51'$
2.7	0.0330	0.1664	0.1696	$78^{\circ}47'$
2.8	0.0224	0.1536	0.1552	$81^{\circ}43'$
2.9	0.0133	0.1414	0.1421	$84^{\circ}39'$
3.0	- 0.0055	0.1299	0.1300	$87^{\circ}34'$
3.1	- 0.0010	0.1190	0.1190	$90^{\circ}30'$
3.2	- 0.0065	0.1088	0.1089	$93^{\circ}25'$

x	Re g	Im g	|g|	arc g
3.3	- 0.0110	0.0991	0.0997	$96^\circ 20'$
3.4	- 0.0147	0.0901	0.0913	$99^\circ 15'$
3.5	- 0.0176	0.0817	0.0836	$102^\circ 10'$
3.6	- 0.0199	0.0739	0.0765	$105^\circ 05'$
3.7	- 0.0216	0.0666	0.0700	$108^\circ 00'$
3.8	- 0.0229	0.0599	0.0641	$110^\circ 55'$
3.9	- 0.0237	0.0537	0.0587	$113^\circ 50'$
4.0	- 0.0242	0.0480	0.0537	$116^\circ 45'$
4.1	- 0.0244	0.0428	0.0492	$119^\circ 40'$
4.2	- 0.0243	0.0380	0.0451	$122^\circ 35'$
4.3	- 0.0240	0.0336	0.0413	$125^\circ 30'$
4.4	- 0.0235	0.0296	0.0378	$128^\circ 25'$
4.5	- 0.0228	0.0260	0.0346	$131^\circ 20'$

III. DIFFRACTION OF RADIO WAVES AROUND THE EARTH'S SURFACE

V. Fock

The problem of the propagation of radio waves around the homogeneous surface of the earth is investigated. The diffraction effects are considered but the influence of the ionosphere is neglected. The aim of the paper is to derive formulas for the wave amplitude as a function of the elevation of the source, its distance from the point of observation (situated on the surface of the earth), of the wave length and of electrical properties of the soil. The main result is the derivation of an expression for the attenuation factor in form of an integral. This expression is valid for all the values of parameters which are of practical interest. In the limiting cases the well-known formulas are obtained: the Weyl—van der Pol formula for illuminated region and the formula which corresponds to the first term in Watson's series for the shaded region (the latter in a slightly corrected form). Essentially new is the investigation of the region of the penumbra (near the line of horizon). Formulas are obtained which give a continuous transition from the illuminated region to the shaded one. Methods for numerical calculations of sums and integrals involved in the problem are elaborated.

INTRODUCTION*

There are many papers devoted to the problem of the diffraction of radio waves around the surface of the earth. A review of more recent investigations may be found in a paper by B. Vvedensky.[3]

The interest in this problem is justified by the fact, that at small distances, of the order of a few hundreds of

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*A short account of the results of this paper is given in our note.[1]

kilometres, the refraction of radio waves in the ionized layers of the atmosphere may be neglected and the decisive role in the propagation of radio waves is played by the diffraction.

In spite of the fact that a rigorous solution of the problem of diffraction by the sphere had been already obtained some decades earlier, no practically suitable approximate solution has been proposed up to now. In this paper we intend to fill up this gap.

1. STATEMENT OF THE PROBLEM AND ITS SOLUTION IN THE FORM OF SERIES

We denote by r, θ, ϕ spherical coordinates with origin at the center of the earth globe.

The equation of the earth's surface (considered as smooth) is $r = a$, where a is the radius of the earth. Let us suppose that a vertical electric dipole is located at the point $r = b$, $\theta = 0$ (where $b>a$). Suppressing the time-dependent factor $e^{-i\omega t}$ in the field components, we can express these components by means of the Hertz function U which depends on r and θ only. Denoting by k the absolute value of the wave vector we obtain for the field in the air:

$$E_r = \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial U}{\partial\theta}\right);$$

$$E_\theta = -\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial U}{\partial\theta}\right); \qquad (1.01)$$

$$H_\phi = -ik\frac{\partial U}{\partial\theta},$$

the other components being equal to zero.

Similar equations hold for the field in the earth.

The function U satisfies, for $r > a$ the equation

$$\Delta U + k^2 U = 0, \qquad (1.02)$$

and the radiation condition at infinity

$$\lim_{r \to \infty} \left(\frac{\partial r U}{\partial r} - i k r\, U \right) = 0. \qquad (1.03)$$

If $b > a$, i. e. if the source (dipole) is located over the earth's surface and not on the surface itself, U must have a singularity at the point $r = b$, $\theta = 0$, such that

$$U = \frac{e^{ikR}}{R} + U^*, \qquad (1.04)$$

and U^* remains finite if $kR \to 0$. In this formula

$$R = \sqrt{r^2 + b^2 - 2rb \cos \theta} \qquad (1.05)$$

is the distance from the dipole. On the earth's surface the Hertz function U has to satisfy the boundary conditions which ensure the continuity of the tangential components E_θ and H_ϕ.

If we denote the Hertz function within the earth by U_2 these boundary conditions will have the form:

$$k^2 U = k_2^2 U_2; \quad \frac{\partial}{\partial r}(rU) = \frac{\partial}{\partial r}(rU_2) \quad \text{for } r = a. \qquad (1.06)$$

For $0 \leqslant r \leqslant a$ (within the earth) the function U_2 has to satisfy an equation similar to (1.02) and to remain finite.

The quantity k_2 in formula (1.06) and in subsequent formulas is determined by the equation

$$k_2^2 = \epsilon k^2 + i \frac{4\pi\sigma}{c} k \qquad (1.07)$$

and by the condition $\mathrm{Im}(k_2) > 0$. It is useful to introduce instead of the conductivity of the earth σ, a length l which characterizes the specific resistance of the earth. We put

$$l = c/4\pi\sigma. \tag{1.08}$$

For sea water the values of l vary from 0.05 cm (very salty water) to 0.5 cm (scarcely salty water). For the soil this length is hundreds or thousands times greater. Introducing the complex inductive capacity of the earth

$$\eta = \epsilon + i\,\frac{\lambda}{2\pi l} \tag{1.09}$$

we have

$$k_2 = k\sqrt{\eta} \tag{1.10}$$

The solution of our problem in the form of series is well known. We write down the necessary formulas, without giving their derivation

$$\begin{aligned} \psi_n(x) &= \sqrt{\frac{\pi x}{2}}\, J_{n+\frac{1}{2}}(x); \\ \zeta_n(x) &= \sqrt{\frac{\pi x}{2}}\, H^{(1)}_{n+\frac{1}{2}}(x), \end{aligned} \tag{1.11}$$

where $J_v(x)$ is the Bessel function and $H_v^{(1)}(x)$ is the Hankel function of the first kind. These functions are connected by the relation

$$\psi_n(x)\,\zeta_n'(x) - \psi_n'(x)\,\zeta_n(x) = i. \tag{1.12}$$

We introduce a special notation for the logarithmic derivative of the function $\psi_n(x)$:

$$\chi_n(x) = \frac{\psi_n'(x)}{\psi_n(x)}. \tag{1.13}$$

As seen from (1.01), the field on the earth's surface may be expressed by the quantities

$$U_a = U\big|_{r=a}; \quad U'_a = \frac{\partial}{\partial r}(rU)\big|_{r=a}. \qquad (1.14)$$

For these quantities the following series in Legendre polynomials may be obtained:

$$U_a = -\frac{1}{kab}\sum_{n=0}^{\infty}\frac{(2n+1)\,\zeta_n(kb)}{\zeta'_n(ka) - \frac{k}{k_2}\,\chi_n(k_2a)\,\zeta_n(ka)}\,P_n(\cos\theta), \qquad (1.15)$$

$$U'_a = -\frac{k}{k_2b}\sum_{n=0}^{\infty}\frac{(2n+1)\,\zeta_n(kb)\,\chi_n(k_2a)}{\zeta'_n(ka) - \frac{k}{k_2}\,\chi_n(k_2a)\,\zeta_n(ka)}\,P_n(\cos\theta). \qquad (1.16)$$

Our task is to perform an approximate summation of these series.

2. THE SUMMATION FORMULA

The sums we have to calculate are of the form

$$S = \sum \nu\phi(\nu)P_{\nu-\frac{1}{2}}(\cos\theta), \qquad (2.01)$$

where the summation is taken over half integral values of ν.

In the sum (1.15) the function $\phi(\nu)$ (disregarding a constant factor) is equal to

$$\phi(\nu) = \frac{\zeta_{\nu-\frac{1}{2}}(kb)}{\zeta'_{\nu-\frac{1}{2}}(ka) - \frac{k}{k_2}\,\chi_{\nu-\frac{1}{2}}(k_2a)\,\zeta_{\nu-\frac{1}{2}}(ka)}. \qquad (2.02)$$

In the sum (1.16) this function differs from (2.02) by the factor $\chi_{\nu-\frac{1}{2}}(k_2a)$.

For the direct computation of the sum it would be necessary to take the number of the terms approximately equal to 2ka, i. e. to double the number of the waves which may be put around the earth circumference. Since this number is enormous, it is evident, that such a direct summation is impossible. For the calculation of the sum S it is necessary to make use of the fact that $\phi(\nu)$ is an analytical function and to transform this sum into an integral, which is to be evaluated by some approximate method. Such a transformation was firstly proposed by Watson[2] in 1918 and was then used by various authors. But all these authors aimed to bring the expression obtained by this transformation to the form of a sum of residues, while our aim is to separate out a main term which is easier to investigate and to estimate the magnitude of the remainder. The method of computation of the main term is not predetermined thereby.

When performing our transformation we have to bear in mind the following general properties of the function $\phi(\nu)$. It is an analytical function of ν meromorphic in the right half-plane. It has poles only in the first quadrant and is holomorphic in the fourth quadrant. It decreases at infinity in such a way that all the integrals considered converge.

The Legendre functions that enter (2.01) can be expressed by means of the function

$$G_\nu = \frac{\Gamma(\nu + \frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\nu + 1)}\, F\left(\tfrac{1}{2}, \tfrac{1}{2}, \nu + 1, \frac{ie^{-i\theta}}{2 \sin \theta}\right) \qquad (2.03)$$

where F denotes the hypergeometrical function. Denoting by G_ν^* and by $P^*_{\nu-\frac{1}{2}}$ the expressions which are obtained from G_ν and from $P_{\nu-\frac{1}{2}} = P_{\nu-\frac{1}{2}}(\cos\theta)$ by replacing θ by $\pi - \theta$ we get:

$$P_{\nu-\frac{1}{2}} = \frac{1}{\pi\sqrt{2\sin\theta}}\left[e^{i\nu\theta-i\frac{\pi}{4}}G_\nu^* + e^{-i\nu\theta+i\frac{\pi}{4}}G_\nu\right]. \qquad (2.04)$$

It is seen from (2.03) that if the values of ν lie outside of a certain sector, which includes the negative real axis, and if $|\nu\sin\theta|$ is large, then the function G_ν (and also G_ν^*) is approximately equal to

$$G_\nu \sim \sqrt{\pi/\nu}. \qquad (2.05)$$

Substituting (2.05) in (2.04) we get the well known asymptotic expression for $P_{\nu-\frac{1}{2}}$. If we denote by $B(\nu)$ the first term in formula (2.04):

$$B(\nu) = \frac{1}{\pi\sqrt{2\sin\theta}}\, e^{i\nu\theta-i\frac{\pi}{4}}\, G_\theta^*, \qquad (2.06)$$

the following relation may be proved

$$P^*_{\nu-\frac{1}{2}} = e^{i(\nu-\frac{1}{2})\pi}\, P_{\nu-\frac{1}{2}} + 2i\cos\nu\pi\, B(\nu). \qquad (2.07)$$

We shall use this relation later on. We note that $B(\nu)$ is holomorphic in the right half-plane.

Let us consider in the plane of the complex variable ν three contours: 1) the loop C_1 which starts at infinity on the positive real axis, runs above the real axis, encircles the origin counter-clockwise and returns to the starting point at infinity running below the real axis; 2) the broken line C_2, which contains the first quadrant and is described (in its horizontal part drawn slightly over the real axis) from the left to the right side; 3) the straight line C_3 which crosses the origin and is inclined at a small angle to the imaginary axis. This line is described from the top to the bottom and lies in the second and fourth quadrants.

We can write the sum S in the form

$$S = \frac{1}{2} \int_{C_1} \nu\phi(\nu) \sec \nu\pi \; P^{*}_{\nu-\frac{1}{2}} \, d\nu, \tag{2.08}$$

since the integral on the right-hand side reduces to the sum of the residues in the points $\nu = n + \frac{1}{2}$. The function $\phi(\nu)$ being holomorphic in the fourth quadrant, we may replace the contour C_1 by the contours C_2 and C_3 and write

$$\begin{aligned} S = &- \frac{1}{2} \int_{C_2} \nu\phi(\nu) \sec \nu\pi \; P^{*}_{\nu-\frac{1}{2}} \, d\nu \; + \\ &+ \frac{1}{2} \int_{C_3} \nu\phi(\nu) \sec \nu\pi \; P^{*}_{\nu-\frac{1}{2}} \, d\nu. \end{aligned} \tag{2.09}$$

This transformation of the sum corresponds to the usual one; the integral along the contour C_3 is neglected because of the smallness of the odd part of $\phi(\nu)$ (an estimate of its magnitude will be given below), and the integral along C_2 is reduced to the sum of residues. But we shall go a step further and divide the integral along C_2 into two parts: the main term and the correction term. Inserting in the integral the expression (2.07) for $P^*_{\nu-\frac{1}{2}}$ we shall have

$$S = S_1 + S_2 + S_3, \tag{2.10}$$

where

$$S_1 = \int_C \nu\phi(\nu)\, B(\nu)\, d\nu, \tag{2.11}$$

$$S_2 = -\frac{1}{2}\int_{C_2} \nu\phi(\nu) \sec \nu\pi\, e^{i\nu\pi}\, P_{\nu-\frac{1}{2}}\, d\nu \tag{2.12}$$

$$S_3 = \frac{1}{2}\int_{C_3} \nu\phi(\nu) \sec \nu\pi\, P^*_{\nu-\frac{1}{2}}\, d\nu. \tag{2.13}$$

The integrand in S_1 has no poles on the real axis (and also in the fourth quadrant). Therefore, there is no difference, whether we evaluate the integral S_1 along C_2 or along C_3. We have denoted by C any contour, which is equivalent to C_2 or C_3.

The representation of S as a sum of three integrals (2.10) is exact—there was made no neglection in our derivation. But the estimation of the magnitude of S_2 and S_3 shows that these integrals are negligibly small as compared to S_1.

In fact, if we evaluate the integral S_2 as a sum of residues at the poles of $\phi(\nu)$ we shall see that its ratio to S_1 is of the order

$$\left| e^{2i\nu_1 (\pi-\theta)} \right| \tag{2.14}$$

where ν_1 is the pole of $\phi(\nu)$ nearest to the real axis. The imaginary part of ν_1 is positive and for large values of ka will be

$$\mathrm{Im}(\nu_1) = c(ka)^{1/3}, \tag{2.15}$$

where c is a pure number of the order of unity (for the perfect conductor c = 0.70). Since ka is very large, of the order of a million (for λ = 40 m, ka = 10^6), it is clear, that the quantity (2.15) will be large (for instance, equal to 70) and the quantity (2.14) will be negligibly small. (In our problem θ cannot reach the value π since in this case we have to take into account the influence of ionized layers of the atmosphere and our formulas cease to be valid.)

The value of the integral S_3 is determined by the odd part of $\phi(\nu)$. But the odd part of this function will be of the order

$$\left| e^{2ik_2a} \right| . \tag{2.16}$$

Since the imaginary part of k_2a is a positive and very large, the value of (2.16) will be inconceivably small.

The following physical picture gives a notion of the smallness of the integrals S_2 and S_3. The integral S_2 is the amplitude of a wave which travelled once or several times around the globe without refraction (by means of diffraction only). The integral S_3' is the amplitude of a wave which traversed a path equal to the diameter of the globe with the absorption which takes place within the earth. It is clear that both the integrals are negligibly small as compared with the amplitude of the wave which reached the observer through the air by the nearest way.

Therefore with the whole permissible accuracy (i.e. with an error which is negligibly small as compared with the errors involved in the position of our physical problem) the sum S defined by (2.01) may be put equal to the integral S_1 alone. This integral may be written in the form

$$S_1 = \frac{e^{-i(\pi/4)}}{\pi\sqrt{2\sin\theta}} \int_C v\phi(v)\, e^{iv\theta}\, G_v^*\, dv, \tag{2.17}$$

which follows from (2.11) when the expression (2.06) for B(v) is inserted.

3. THE EVALUATION OF THE HERTZ FUNCTION FOR THE ILLUMINATED REGION

If $\phi(v)$ is the function (2.02), then the relation between the sum S and the quantity U_a is

$$U_a = -\frac{2}{kab} S. \tag{3.01}$$

Therefore, our approximate expression for U_a may be written

$$U_a = \frac{2e^{i\frac{3\pi}{4}}}{\pi\, kab\sqrt{2\sin\theta}} \int_C v\phi(v)\, e^{iv\theta}\, G_v^*\, dv. \tag{3.02}$$

The position of the main part of the integration path in (3.02) depends on the point for which the integral is evaluated. In general the main part is in the vicinity of the point $\nu = \nu_o$, where

$$\nu_o = kh_c = k \frac{ab \sin \theta}{\sqrt{a^2 + b^2 - 2ab \cos \theta}} \tag{3.03}$$

The quantity h_c is the length of the perpendicular dropped from the earth's center on the ray (i. e., on the straight line which connects the source and the point of observation).

For the approximate evaluation of the integral U_a it is necessary to obtain the asymptotic expressions for the functions G_ν^* and $\phi(\nu)$ valid on the main part of the integration path. Since ν_o and $\nu_o\theta$ are large as compared with unity, we may put according to (2.05)

$$G_\nu^* = \sqrt{\pi/\nu}. \tag{3.04}$$

For the Hankel functions involved in $\phi(\nu)$ one may tentatively use the Debye expression

$$\zeta_{\nu-\frac{1}{2}}(\rho) = \frac{1}{\sqrt[4]{1 - (\nu^2/\rho^2)}} e^{i\left(\xi - \frac{\pi}{4}\right)}, \tag{3.05}$$

where

$$\xi = \int_\nu^\rho \sqrt{1 - \frac{\nu^2}{\rho^2}}\, d\rho. \tag{3.06}$$

These expressions are valid provided the condition

$$|\rho^2 - \nu^2| >> \rho^{4/3} \tag{3.07}$$

is satisfied. As to the function $\chi_{\nu-\frac{1}{2}}(k_2a)$ its value near the point $\nu = \nu_0$ may be represented with a sufficient approximation by the expression

$$\chi_{\nu-\frac{1}{2}}(k_2a) = -i\sqrt{1-\frac{\nu^2}{k_2^2\,a^2}} \tag{3.08}$$

In order to make clear, in which cases the inequality (3.07) is satisfied, let us introduce the parameter

$$p = \left(\frac{ka}{2}\right)^{1/3}\cos\gamma, \tag{3.09}$$

where γ is the angle between the vertical direction at the observation point and the direction from this point to the source.

It is easily seen that for $\nu = \nu_0$, $\rho = ka$ the inequality (3.07) is equivalent to the condition that p should be large and positive. Such values of p correspond to the illuminated region. The values of p of the order of unity (positive and negative ones) correspond to the region of penumbra: the special value p = 0 gives the boundary of the geometrical shadow (horizon line). Large and negative values of p correspond to the shadow region.

In this section we shall investigate the case of a large positive p (illuminated region); other cases will be investigated in the next sections.

We have seen that if $p \gg 1$ the Debye expressions for the Hankel functions are valid. Inserting these expressions into

(3.02) and using (3.04) and (3.08) we get

$$U_a = \frac{2e^{i\frac{\pi}{4}}}{kab\sqrt{2\pi \sin\theta}}\sqrt{\frac{b}{a}}\int_C \sqrt[4]{\frac{k^2a^2 - \nu^2}{k^2b^2 - \nu^2}} \cdot$$

$$\cdot \frac{e^{i\omega}\sqrt{\nu}\, d\nu}{\sqrt{1 - \frac{\nu^2}{k^2a^2}} + \frac{k}{k_2}\sqrt{1 - \frac{\nu^2}{k_2^2a^2}}}, \tag{3.10}$$

where

$$\omega = \int_{ka}^{kb}\sqrt{1 - \frac{\nu^2}{\rho^2}}\, d\rho + \nu\theta. \tag{3.11}$$

If the condition

$$kh \cos\gamma \gg 1 \tag{3.12}$$

is satisfied, where $h = b - a$ is the height of the source above the earth, the integral (3.10) can be calculated by means of the method of the steepest descent and the following "reflection formula" is obtained:

$$U_a = \frac{e^{ikR}}{R} W. \tag{3.13}$$

In this formula

$$R = \sqrt{a^2 + b^2 - 2ab\cos\theta} \tag{3.14}$$

is the distance from the source, and W is the "attenuation function" which in our case is equal to

$$W = \frac{2}{1 + \frac{k}{k_2}\sqrt{1 - \frac{k^2}{k_2^2}\sin^2\gamma} \cdot \sec\gamma}. \tag{3.15}$$

The quantity U_a' defined by the series (1.16) differs (in our approximation) from U_a by a constant factor only. We have

$$U_a' = -\frac{ik^2a}{k_2}\sqrt{1 - \frac{k^2}{k_2}\sin^2\gamma}\; U_a. \tag{3.16}$$

The last formula is true not only for the illuminated region, but also in other cases.

If condition (3.12) is not satisfied, the denominator in the integrand (3.10) cannot be considered as slowly varying. If instead of (3.12) we suppose that the conditions:

$$1 \ll \frac{R^2}{h^2} \ll (ka)^{2/3}, \tag{3.17}$$

$$1 \ll kR \ll a/h, \tag{3.18}$$

are satisfied (the inequality $p \gg 1$, being a consequence of these conditions), the integral (3.10) can be approximately calculated by introducing a new integration variable μ, according to

$$\mu = \sqrt{1 - \frac{\nu^2}{k^2a^2}} \; . \tag{3.19}$$

For the function W in (3.13) the following approximate expression is obtained:

$$W = e^{-i\frac{3\pi}{4}}\sqrt{\frac{2kR}{\pi}}\int_{\Gamma} e^{-i\frac{kR}{2}(\mu - \mu_o)^2}\,\frac{\mu\, d\mu}{\mu + (k/k_2)} , \tag{3.20}$$

where

$$\mu_o = h/R \tag{3.21}$$

is the inclination of the ray to the horizon. The contour Γ is a straight line which crosses the point $\mu = \mu_o$ passing there from the fourth to the second quadrant of the plane of μ (or of $\mu - \mu_o$ to be more exact). The integral (3.20) can be calculated without any further approximation and gives the well-known Weyl– van der Pol formula.

If we put

$$\sigma = e^{i\frac{\pi}{4}} \frac{k}{k_2} \sqrt{\frac{kR}{2}}, \quad \tau = e^{i\frac{\pi}{4}} \frac{h}{R} \sqrt{\frac{kR}{2}}, \tag{3.22}$$

we shall have

$$W = 2 - 4\sigma e^{-(\sigma+\tau)^2} \int_{i\infty}^{\sigma+\tau} e^{a^2}\, da. \tag{3.23}$$

To obtain the field components from our expressions for U_a and U_a' we have to differentiate these expressions by θ which is easily done, since we may regard all factors in (3.13) except e^{ikR}, as constants.

4. ASYMPTOTIC EXPRESSIONS FOR THE HANKEL FUNCTIONS

In the following we have to consider the case when the point of observation is in the region of penumbra.

This case is characterized by the values of the parameter p (positives or negatives) of the order unity. As the inequality

(3.01) is not satisfied in this case, the Debye expressions (3.05) for the Hankel functions are not valid on the main part of the integration contour and must be replaced by some others. The new expressions for the Hankel functions suitable for our purpose can be obtained from the asymptotic expressions which are given in our previous paper[3], or from the formulas given in the well-known Watson's treatise[4], but it is more simple to deduce them independently.

Our aim is to find an approximate expression for the Hankel function in terms of the function w(t), defined by the integral

$$w(t) = \frac{1}{\sqrt{\pi}} \int_{\Gamma} e^{tz-1/3z^3}\, dz, \qquad (4.01)$$

the contour Γ running from infinity to the origin along the ray arc $z = -2\pi/3$ and from the origin to infinity along the ray arc $z = 0$ (the positive real axis). The function w(t) satisfies the differential equation

$$w''(t) = tw(t) \qquad (4.02)$$

with the initial conditions:

$$w(0) = \frac{2\sqrt{\pi}}{3^{2/3}\,\Gamma(2/3)}\, e^{i(\pi/6)} = 1.0899290710 + i0.6292708425,$$

$$w'(0) = \frac{2\sqrt{\pi}}{3^{4/3}\,\Gamma(4/3)}\, e^{-i(\pi/6)} = 0.7945704238 - i0.4587454481. \qquad (4.03)$$

w(t) is an integral transcendental function, which can be expanded into a power series of the form:

$$w(t) = w(0)\left\{1 + \frac{t^3}{2\cdot 3} + \frac{t^6}{(2\cdot 5)(3\cdot 6)} + \frac{t^9}{(2\cdot 5\cdot 8)(3\cdot 6\cdot 9)} + \cdots\right\} +$$

$$+ \; w'(0)\left\{t + \frac{t^4}{3\cdot 4} + \frac{t^7}{(3\cdot 6)(4\cdot 7)} + \frac{t^{10}}{(3\cdot 6\cdot 9)(4\cdot 7\cdot 10)} + \cdots\right\}. \tag{4.04}$$

If we separate in w(t) the real and the imaginary parts (for real values of t) putting

$$w(t) = u(t) + iv(t), \tag{4.05}$$

then u(t) and v(t) will be two independent integrals of equation (4.02) connected by the relation

$$u'(t)\, v(t) - u(t)\, v'(t) = 1. \tag{4.06}$$

The asymptotic expressions of these functions for large negative values of t are obtained by separation of the real and imaginary parts in the formulas:

$$w(t) = e^{i\frac{\pi}{4}}\,(-t)^{-1/4}\, e^{i\frac{2}{3}(-t)^{3/2}}, \tag{4.07}$$

$$w'(t) = e^{-i\frac{\pi}{4}}\,(-t)^{1/4}\, e^{i\frac{2}{3}(-t)^{3/2}}. \tag{4.08}$$

For large positive values of t the asymptotic expressions for u(t), v(t) and their derivatives are of the form

$$u\,(t) = t^{-1/4}\, e^{\frac{2}{3}t^{3/2}}; \quad u'(t) = t^{1/4}\, e^{\frac{2}{3}t^{3/2}}; \tag{4.09}$$

$$v(t) = \frac{1}{2}t^{-1/4}\, e^{-\frac{2}{3}t^{3/2}} ;$$

$$v'(t) = -\frac{1}{2}\, t^{1/4}\, e^{-\frac{2}{3}t^{3/2}} . \tag{4.10}$$

From the series (4.04) the following relations are easily deduced:

$$w(te^{i\frac{\pi}{3}}) = 2e^{i\frac{\pi}{6}}\, v(-t), \tag{4.11}$$

$$w(te^{i\frac{2\pi}{3}}) = e^{i\frac{\pi}{3}}\left[u(t) - iv(t)\right] . \tag{4.12}$$

These relations describe the behavior of $w(t)$ in the complex t-plane.

We note that $w(t)$ is expressible in terms of the Hankel function of the order 1/3 according to the formula

$$w(t) = \sqrt{\frac{\pi}{3}}\, e^{i\frac{2\pi}{3}}\, (-t)^{1/2}\, H^{(1)}_{1/3}\left(\frac{2}{3}\,(-t)^{3/2}\right). \tag{4.13}$$

After having enumerated the main properties of $w(t)$, we now proceed to deduce the asymptotic expression for the Hankel function $H^{(1)}_{\nu}(\rho)$ where ν and ρ are large and nearly equal, so that the ratio

$$\frac{\nu - \rho}{\sqrt[3]{\rho/2}} = t \tag{4.14}$$

remains bounded, while ρ tends to infinity.

The Hankel function $H^{(1)}_{\nu}(\rho)$ admits the integral representation

$$H_\nu^{(1)}(\rho) = \frac{1}{\pi i}\int_C e^{-\rho \operatorname{sh} v + \nu v}\, dv, \tag{4.15}$$

where the contour C consists of a part of the straight line $\mathrm{Im}(v) = -\pi$ described from $-\pi i - \infty$ to some point $v = v_0$ with $\mathrm{Re}(v_0) < 0$ $\left[\text{e. g. } v_0 = (-\pi/\sqrt{3}) - i\pi\right]$, a straight line joining v_0 to the origin and, finally, the positive real axis described from the origin to infinity.

Let us express ν through t, according to (4.14), and introduce a new integration variable

$$z = \sqrt[3]{\rho/2}\cdot v. \tag{4.16}$$

Considering t and z as finite and ρ as large, we can expand the intergrand in (4.15) in a series of negative (fractional) powers of ρ. Since the relevant part of the transformed contour C coincides with contour Γ we can write

$$H_\nu^{(1)}(\rho) = \frac{1}{\pi i}\left(\frac{\rho}{2}\right)^{-1/3}\int_\Gamma e^{tz-1/3z^3}\left[1 - \frac{1}{60}\left(\frac{\rho}{2}\right)^{-2/3} z^5 + \ldots\right]dz \tag{4.17}$$

and evaluate the integral using (4.01). We thus obtain

$$H_\nu^{(1)}(\rho) = -\frac{1}{\sqrt{\pi}}\left(\frac{\rho}{2}\right)^{-1/3}\left\{w(t) - \frac{1}{60}\left(\frac{\rho}{2}\right)^{-2/3} w^{(5)}(t) + \ldots\right\}. \tag{4.18}$$

In virtue of the differential equation (4.02) the fifth derivative equals

$$w^{(5)}(t) = t^2 w'(t) + 4tw(t). \tag{4.19}$$

Inserting this in (4.18) and using (1.11) we get the following expression for the function $\zeta_\nu{-1/2}(\rho)$:

$$\zeta_\nu{-1/2}(\rho) = -\left(\frac{\rho}{2}\right)^{1/6}\left\{w(t) - \frac{1}{60}\left(\frac{\rho}{2}\right)^{-2/3}\left[t^2 w'(t) + 4tw(t)\right] + \ldots\right\}. \quad (4.20)$$

Differentiating this expression with respect to ρ (with account of the dependence of t on ρ with ν constant) we get the following expression for the derivative:

$$\zeta'_\nu{-1/2}(\rho) = i\left(\frac{\rho}{2}\right)^{-1/6}\left\{w'(t) - \frac{1}{60}\left(\frac{\rho}{2}\right)^{-2/3}\left[(t^2+9)w(t) - 4tw'(t)\right] + \ldots\right\}. \quad (4.21)$$

These expressions will be used in the next section.

§ 5. The expressions of the Herz function valid in the penumbra region.

We rewrite the expression (3.02) for the Herz function replacing therein the quantity G_ν^* by its approximate value $\sqrt{\pi/\nu}$ and the quantity $\sin\theta$ before the integral by θ. We get

$$U_a = \frac{2e^{i\frac{3\pi}{4}}}{kab\sqrt{2\pi\theta}} \int_C \phi(\nu) e^{i\nu\theta} \sqrt{\nu}\, d\nu. \quad (5.01)$$

The contour C may be taken identical with contour C_2, which was defined in § 2, or may be replaced by some contour equivalent to C_2. The main part of the integration path lies in our case (i.e. for finite values of the parameter p) near the point $\nu = ka$. Consequently, the function $\chi_{\nu-\frac{1}{2}}(k_2 a)$ involved in (2.02) can be replaced by the value of (3.08) for $\nu = ka$. Introducing this in $\phi(\nu)$ we obtain:

$$\phi(\nu) = \frac{\zeta_{\nu-\frac{1}{2}}(kb)}{\zeta'_{\nu-\frac{1}{2}}(ka) + i\,\frac{k}{k_2}\sqrt{1 - \frac{k^2}{k_2^2}}\;\zeta_{\nu-\frac{1}{2}}(ka)} \tag{5.02}$$

For $\zeta_{\nu-\frac{1}{2}}$ and its derivative we must use expressions valid near the point $\nu = ka$. Such expressions were obtained in the preceeding paragraph. Retaining in (4.20) and (4.21) the principal terms only we get:

$$\zeta_{\nu-\frac{1}{2}}(ka) = -\,i\left(\frac{ka}{2}\right)^{1/6} w(t), \tag{5.03}$$

$$\zeta_{\nu-\frac{1}{2}}(ka) = i\left(\frac{ka}{2}\right)^{-1/6} w'(t), \tag{5.04}$$

where the variable t is connected with ν by the relation

$$\nu = ka + \left(\frac{ka}{2}\right)^{1/3} t. \tag{5.05}$$

The numerator in (5.02) is obtained from (5.03) by replacing a by b and t by t', where

$$\nu = kb + \left(\frac{kb}{2}\right)^{1/3} t'. \tag{5.06}$$

Equating (5.05) and (5.06) we obtain the connection between t and t'. Since the ratio h/a, where $h = b - a$, is small [we shall consider it of the same order as $(ka)^{-2/3}$] we must neglect it as compared to unity. We may then put

$$t' = t - y, \tag{5.07}$$

where

$$y = \frac{kh}{(ka/2)^{1/3}} \tag{5.08}$$

is a quantity proportional to the height of the source over the earth's surface. We may call y the reduced height of the source. Hence, with neglect of terms of the order h/a or $(ka)^{-2/3}$ we have:

$$\zeta_{\nu-\frac{1}{2}}(kb) = - i \left(\frac{ka}{2}\right)^{1/6} w(t - y), \tag{5.09}$$

where t is determined by (5.05). (We have also replaced b by a in the factor before w.)

Substitution of (5.03), (5.04) and (5.09) in (5.02) gives the desired approximate expression for $\phi(\nu)$.

If we put for the sake of brevity

$$q = i \left(\frac{ka}{2}\right)^{1/3} \frac{k}{k_2} \sqrt{1 - \frac{k^2}{k_2^2}} \;, \tag{5.10}$$

we obtain

$$\phi(\nu) = - \left(\frac{ka}{2}\right)^{1/3} \frac{w(t - y)}{w'(t) - q\, w(t)} \;. \tag{5.11}$$

Remembering formulas (1.09) and (1.10), we may write for the quantity q

$$q = i \left(\frac{\pi a}{\lambda}\right)^{1/3} \frac{\sqrt{\epsilon - 1 + i(\lambda/2\pi\ell)}}{\epsilon + i(\lambda/2\pi\ell)} \tag{5.12}$$

or with the same accuracy

$$q = i\left(\frac{\pi a}{\lambda}\right)^{1/3} \frac{1}{\sqrt{\epsilon + 1 + i(\lambda/2\pi\ell)}} \tag{5.13}$$

This form is slightly more convenient for calculations.

We have now to substitute the value of $\phi(v)$ from (5.11) into (5.01) and introduce the integration variable t. Making this substitution, we may replace the quantity $\sqrt{v}$ in the integrand by the constant value $\sqrt{ka}$ and also write b instead at a in the factor before the integral. The resulting formula may be written in the form:

$$U_a = \frac{e^{tka\theta}}{a\theta} e^{-i\frac{\pi}{4}} \sqrt{\frac{x}{\pi}} \int_C e^{ixt} \frac{w(t-y)}{w'(t) - qw(t)}\, dt, \tag{5.14}$$

where x denotes the quantity

$$x = \left(\frac{ka}{2}\right)^{1/3} \theta, \tag{5.15}$$

which may be termed as the reduced horizontal distance from the source, while y and q have the values given by (5.08) and (5.13). The contour C must be such that all the poles of the integrand are comprised within the contour; as we shall see later, they are all situated in the first quadrant of the t plane. Thus we can carry out the integration in (5.14) from $i\infty$ to 0 and from 0 to $+\infty$.

In order to get a more clear idea on the ratio of the horizontal and the vertical scale in the variables x and y, we write the expression for the parameter p, as defined by

(3.09), in terms of x and y. From the consideration of the triangle with vertices in the earth's center, in the source point and in the point of observation, the following approximate expression is easily deduced:

$$p = \left(\frac{ka}{2}\right)^{1/3} \cos\gamma = \frac{y - x^2}{2x} . \tag{5.16}$$

It follows that the equation of the horizon line is $x = \sqrt{y}$. Further we shall need the relation between the distance R from the source as measured along a straight line and the horizontal distance $a\theta$ as measured along the arc of a great circle. Assuming $a\theta >> h$, i. e. $(ka)^{1/3}\, x >> y$, this relation may be written

$$kR = ka\theta + \omega_o, \tag{5.17}$$

where

$$\omega_o = \frac{y^2}{4x} + \frac{xy}{2} - \frac{x^3}{12} . \tag{5.18}$$

6. DISCUSSION OF THE EXPRESSION FOR THE HERTZ FUNCTION

The expression obtained for the Hertz function is most conveniently written in the form:

$$U_a = \frac{e^{ika\theta}}{a\theta}\, V\,(x,y,q), \tag{6.01}$$

where

$$V(x,y,q) = e^{-i\frac{\pi}{4}}\sqrt{\frac{x}{\pi}} \int_C \frac{e^{ixt}\, w(t - y)}{w'(t) - qw(t)}\, dt . \tag{6.02}$$

The quantity V may be called attenuation factor by analogy with the quantity W, which was introduced earlier [see (3.13)]. Let us determine the connection between V and W. Since in the denominators of expressions (3.13) and (6.01) the quantities R and $a\theta$ can be considered as equal, it follows from (5.17)

$$W = Ve^{-i\omega_0}. \tag{6.03}$$

We have now to investigate the expression (6.02) for V. We shall first consider the case of large positive values of p (illuminated region). This case has been already discussed by another method (§ 3). But, as formula (6.02) was obtained for the case of a finite p, it seems to be of interest to verify that it is also valid in the case of a large p. If $p \gg 1$, the integration path may be deformed so as to cross the point where $\sqrt{-t} = p$. Its main part will be situated in the domain of large negative values of t, where expressions (4.07) and (4.08) for w and w' are applicable. Using them and applying the method of the steepest descent, we obtain

$$V = e^{i\omega_0} \frac{2}{1 - i(q/p)}, \tag{6.04}$$

and in virtue of (6.03)

$$W = \frac{2}{1 - i(q/p)}. \tag{6.05}$$

The latter expression practically coincides with (3.15). We note that in the case when x is of the order of unity or large the condition $p \gg 1$ is sufficient for the applicability of the method of steepest descent. If x is small, the further condition $y^2 \gg 2x$ is necessary. If the latter condition is not satisfied but the inequality

$$x \ll y \ll 1/x \tag{6.06}$$

is satisfied instead, the integral can be calculated by another method. Further simplifications in the asymptotic expression for $w(t - y)$ can be then made, and the integral (6.02) reduces to the form

$$V = e^{i\frac{\pi}{4}}\sqrt{\frac{x}{\pi}}\int_C \frac{e^{ixt+iy\sqrt{-t}}}{\sqrt{-t}-iq}\,dt. \tag{6.07}$$

Taking $\sqrt{-t}$ as integration variable, we are led to an integral of the form (3.20) $\left[\text{with } \sqrt{-t} = (ka/2)^{1/3}\mu\right]$ and we get again the Weyl-van der Pol formula (3.23) with the following values of σ and τ:

$$\sigma = e^{-i\frac{\pi}{4}} q\sqrt{x}, \qquad \tau = e^{i\frac{\pi}{4}}\frac{y}{2\sqrt{x}}. \tag{6.08}$$

These values practically coincide with (3.22).

Let us now investigate the most interesting case when p is of the order of unity (positive or negative). We know that this is the region of the penumbra, where the diffraction effects play the dominant part.

If the values of x and y are of the order of unity, the most effective method of evaluation of the integral (6.02) is the representation of this integral in form of a sum of residues taken at the poles of the integrand.

Denoting by $t_s = t_s(q)$ the roots of the equation

$$w'(t) - qw(t) = 0 \tag{6.09}$$

we obtain

$$V(x,y,q) = e^{i\frac{\pi}{4}} 2\pi x \sum_{s-1}^{\infty} \frac{e^{ixt_s}}{t_s - q^2} \frac{w(t_s - y)}{w(t_s)} . \tag{6.10}$$

The roots $t_s(q)$ are functions of the complex parameter q. For the value $q = 0$ they reduce to the roots $t_s' = t_s(0)$ of the derivative $w'(t)$ and for $q = \infty$ they reduce to the roots $t_s^o = t_s(\infty)$ of the function $w(t)$. The phases of t_s' and t_s^o are equal to $\pi/3$, so that

$$t_s' = \left| t_s' \right| e^{i\frac{\pi}{3}}; \qquad t_s^o = \left| t_s^o \right| e^{i\frac{\pi}{3}}. \tag{6.11}$$

We give here the moduli of the first five roots t_s' and t_s^o:

s	t_s'	t_s^o
1	1.01879	2.33811
2	3.24820	4.08795
3	4.82010	5.52056
4	6.16331	6.78671
5	7.37218	7.99417

For large values of s we have approximately

$$\left| t_s' \right| \cong \left[\frac{3\pi}{2}\left(s - \frac{3}{4}\right)\right]^{2/3},$$

$$\left| t_s^o \right| \cong \left[\frac{3\pi}{2}\left(s - \frac{1}{4}\right)\right]^{2/3}. \tag{6.12}$$

To calculate the roots for finite values of q we may use the differential equation

$$\frac{dt_s}{dq} = \frac{1}{t_s - q^2} . \tag{6.13}$$

which can be easily derived from (4.02). The root $t_s(q)$ is determined either as that solution of (6.13) which at $q = 0$ reduces to t_s' or as that solution which at $q = \infty$ reduces to t_s^o. Both definitions are equivalent. Starting from the first definition, a series in ascending powers of q may be easily constructed for t_s; this series will converge for $|q| < |\sqrt{t_s}|$. Starting from the second definition we may construct a series in descending (negative) powers of q; this will converge for $|q| > |\sqrt{t_s}|$. These series shall not be written down here. It may be noticed that the value of t, which for large values of $|q|$ is close to q^2, is not a root of equation (6.09).

If the condition $y^2 << 2|\sqrt{t_s}|$ is satisfied, we have the approximate relation

$$\frac{w(t_s - y)}{w(t_s)} = ch(y\sqrt{t_s}) - \frac{q}{\sqrt{t_s}} sh(y\sqrt{t_s}) . \tag{6.14}$$

This relation permits us to estimate the value of remote terms in the series (6.10). If s is so large that $|q| << |\sqrt{t_s}|$, we have approximately $t_s = t_s(0) = t_s'$. It follows from this and from expression (6.14) that the series (6.10) is always convergent. But if x is small or if y is large, the series converges slowly, and to calculate its sum a large number of terms may be required.

In the shadow region, where p is large and negative, the series (6.10) converges very rapidly and its sum approximately reduces to its first term.

Our series (6.10) corresponds to that of Watson but has the advantage of simplicity.

The fundamental formula (6.02) permits us to investigate not only the limiting cases (large positive values of p-illuminated region, large negative values of p-shadow region) but also the intermediate cases, namely the region of the penumbra. While in the limiting cases our formula leads to an improvement of formulas previously known (the reflection formula and the Weyl-van der Pol formula for the illuminated region and the Watson series for the shadow region), in the transitional penumbra region it yields essentially new results.

The case when x and y are large and p-finite (short waves, penumbra) is of special interest. This case has not been investigated before as the known formulas are not valid here. In what follows we shall derive approximate formulas, which allow a complete discussion of this case.

We introduce the quantity

$$z = x - \sqrt{y}, \tag{6.15}$$

which represents the reduced distance measured from the boundary of the geometrical shadow (and not from the source). In the region of geometrical shadow we have $z > 0$, in the visible region $z < 0$. Our parameter p, expressed in terms of z and x, takes the form

$$p = \frac{y - x^2}{2x} = - z + \frac{z^2}{2x} \tag{6.16}$$

In our case x is large an z is finite; hence we have approximately $p = - z$.

The main part of the integration path in (6.02) corresponds now to values of t of the order of unity; but if y is large and t finite we may use for $w(t-y)$ the asymptotic expression (4.07) which gives

$$w(t - y) = e^{i\frac{\pi}{4}} (y - t)^{-1/4} e^{i\frac{2}{3}(y-t)^{3/2}} \tag{6.17}$$

or approximately

$$w(t - y) = e^{i\frac{\pi}{4}} y^{-1/4} e^{i\frac{2}{3} y^{3/2} - i\sqrt{y}t}. \tag{6.18}$$

Inserting (6.18) into (6.02) and replacing in the factor before the integral the quantity $x^{\frac{1}{2}} y^{-\frac{1}{4}}$ by unity, we get

$$V(x,y,q) = e^{i\frac{2}{3} y^{3/2}} V_1(x - \sqrt{y},q), \tag{6.19}$$

where

$$V_1(z,q) = \frac{1}{\sqrt{\pi}} \int_C \frac{e^{izt}}{w'(t) - qw(t)} dt. \tag{6.20}$$

The terms neglected in (6.19) are (for a finite z) of the order of $1/\sqrt{y}$ (or of $1/x$).

Therefore, the function $V(x,y,q)$ of two arguments x,y and of the parameter q reduces in our case to a function $V_1(z,q)$ of a single argument z and of the same parameter q. The resulting simplification is quite essential.

Let us now derive the relation connecting the attenuation function W with the function V_1. We have the identity

$$\frac{2}{3} y^{3/2} = \omega_o + \frac{1}{3} z^3 - \frac{z^4}{4x}, \tag{6.21}$$

where ω_o has the value (5.18). Omitting in (6.21) the last term we obtain from (6.03) and (6.19)

$$W = e^{\frac{1}{3} z^3} V_1(z,q). \tag{6.22}$$

Thus, in our approximation function W depends on x and y only through $z = x - \sqrt{y}$.

The function $V_1(z,q)$ is an integral transcendental function of the variable z. For a positive z we can evaluate the integral (6.20) as a sum of residues, and we get

$$V_1(z,q) = i2\sqrt{\pi} \sum_{s=1}^{\infty} \frac{e^{izt_s}}{(t_s - q^2)\, w(t_s)} \tag{6.23}$$

$$(\text{for } z > 0),$$

where t_s are the roots of equation (6.09) which were discussed earlier. The larger is z the more rapidly converges the series (6.23). For a sufficiently large positive z its sum reduces to the first term. For finite negative values of z(e.g. $-2 < z < 0$) the integral (6.20) has to be evaluated by quadratures.

For large negative values of z this integral may be evaluated by the method of steepest descent, and we get

$$V_1(z,q) = \frac{2e^{-(1/3)z^3}}{1 + (iq/z)} . \tag{6.24}$$

According to (6.22), this gives

$$W = 2/\left(1 + \frac{iq}{z}\right). \tag{6.25}$$

Since approximately $z = -p$, this coincides with expression (6.05).

We note in conclusion that our fundamental formula (6.02) can be obtained by the method of parabolic equation, proposed by M. Leontovich and applied by him[5] to the derivation of the Weyl-van der Pol formula. The application of Leontovich's method (in a slightly improved form) to our problem will be given in a separate paper.

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IV. SOLUTION OF THE PROBLEM OF PROPAGATION OF ELECTROMAGNETIC WAVES ALONG THE EARTH'S SURFACE BY THE METHOD OF PARABOLIC EQUATION

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The problem of propagation of electromagnetic waves along the surface of the earth is solved by the method of parabolic equation proposed by Leontovich. In the first section the surface of the earth is considered as plane and the well-known Weyl-van der Pol formula is deduced. This formula turns out to be the exact solution of the parabolic equation with corresponding boundary conditions. In the second section the surface is considered as spherical, and the resulting formula coincides with that obtained by Fock by the method of summation of infinite series representing the rigorous solution of the problem.

A new form of the solution of the problem of propagation of electromagnetic waves from a vertical elementary dipole situated at a given height above the spherical surface of the earth was given in a paper by Fock (1,2)*. In this solution the field is calculated for points on the surface of the earth, but according to the reciprocity theorem the same solution gives directly the field at any point above the surface if the dipole is located on the surface itself. In the present paper it is shown that Fock's solution can also be obtained by another method, namely by reducing the problem to an equation of parabolic type for the "attenuation function".

*In the sequel these papers will be referred as I.

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The method of parabolic equation was proposed by Leontovich and applied by him to the solution of the same problem for the case of a plane earth. Since the considerations of the original paper by Leontovich (3)** need some modifications, we shall give in what follows a new exposition of the method, applying it firstly to the case of a plane earth and considering then the case of a spherical earth.

1. THE CASE OF A PLANE EARTH

We assume the time-dependence of all the field components to be of the form $e^{-i\omega t}$. In the following this factor shall be omitted.

Let us denote by k the absolute value of the wave vector and by η the complex inductive capacity of the earth:

$$k = \frac{2\pi}{\lambda} \; ; \; \eta = \epsilon + i \, \frac{4\pi\sigma}{\omega} = \epsilon + \frac{i}{k\ell} \; . \tag{1.01}$$

The quantity

$$\ell = \frac{c}{4\pi\sigma} \tag{1.02}$$

having the dimensions of a length characterizes the specific resistance of the earth (this length varies from some tenths of a centimeter for sea water to ten and more meters for dry soil). Let U be the vertical component of the Hertz vector (the Hertz function). This function satisfies the equation

$$\Delta U + k^2 U = 0 \; . \tag{1.03}$$

We shall write the Hertz function in the form

$$U = \frac{e^{ikR}}{R} \; W \; , \tag{1.04}$$

**This pap - will be referred in the sequel as II.

where R is the distance from the point of observation to the source and the factor W is the so-called "attenuation function". As it is known, for $kR \to 0$ the Hertz function tends to infinity in such a way that W takes a finite value. We normalize W in such a manner that this value shall be equal to unity (it being supposed that both the source and the observation point remain above the surface of the earth).

In the following we assume, however, that the source is located on the earth's surface. Let us introduce cylindrical coordinates r, z with the origin in the dipole and the z axis drawn vertically upwards. On the earth's surface we have $z = 0$. The distance R will be $R = \sqrt{r^2 + z^2}$. The principal "large parameter" of our problem is the quantity $|\eta|$. For large $|\eta|$ the attenuation function W is a slowly varying function of coordinates. In order to characterize the slowness of its variation it is useful to introduce the dimensionless coordinates:

$$\rho = \frac{kr}{2|\eta|} \quad ; \quad \zeta = \frac{kz}{\sqrt{|\eta|}} \quad , \tag{1.05}$$

and to consider W as a function of ρ and ζ. The derivatives of W with respect to its arguments will be then of the same order of magnitude as the function W itself.

Substitution of (1.04) into equation (1.03) gives for the function $W(\rho,\zeta)$ an equation, which can be simplified if one supposes that the inclination angle of the ray to the horizon is small and that the distance from the source is at least equal to several wave lengths. These assumptions yield the inequalities:

$$\frac{z}{r} \ll 1 \; ; \; kR \gg 1 \; , \tag{1.06}$$

which are equivalent to

$$\frac{\zeta}{\rho} \ll 2\sqrt{|\eta|} \quad ; \quad \rho \gg \frac{1}{2|\eta|} \quad . \tag{1.07}$$

Since $|\eta|$ is assumed to be large, the inequalities (1.07) hold in a wide range of the values of ρ and ζ (and in any case for values of ρ and ζ of the order of unity). If the inequalities (1.07) are valid, the equation for $W(\rho,\zeta)$ assumes the form

$$\frac{\partial^2 W}{\partial \zeta^2} + i\left(\frac{\partial W}{\partial \rho} + \frac{\zeta}{\rho}\frac{\partial W}{\partial \zeta}\right) = 0 . \tag{1.08}$$

The terms omitted in (1.08) are of the order of $1/|\eta|$ as compared with those retained.

The boundary condition for W on the earth's surface is obtained from the condition for the Hertz vector

$$\frac{\partial U}{\partial z} = -\frac{ik}{\sqrt{\eta}}\, U \quad (\text{for } z = 0) \tag{1.09}$$

given by Leontovich. It has the form

$$\frac{\partial W}{\partial \zeta} + q_1 W = 0 \quad (\text{for } \zeta = 0) \tag{1.10}$$

where

$$q_1 = i\sqrt{\frac{|\eta|}{\eta}}\; e^{i\,\frac{\pi-\delta}{2}} \tag{1.11}$$

and δ is the so-called loss angle, defined by

$$\delta = \text{arc tg}\,\frac{1}{k\ell\epsilon}\;;\quad 0 < \delta < \frac{\pi}{2} . \tag{1.12}$$

In the limit $|\eta| \to \infty$ the range of the variations of ρ and ζ is $0 < \rho < \infty$, $0 < \zeta < \infty$.

As a "condition at infinity" we may require that for all positive values of ρ and ζ [with the possible exception of the singular point $\rho = 0$ of equation (1.08)] the function W should be bounded or such that the Hertz vector U is bounded.

We now proceed to the formulation of the condition for $\rho = 0$. Since this is a point of some delicacy, we shall discuss it in a more detailed way.

We must state, firstly, that in the region close to the source, i.e., for small values of kR, the inequalities (1.07) cease to be satisfied; the differential equation (1.08) and the expression for W to be deduced from it become invalid. The region of small kR is a "forbidden zone" for our approximate function W. Therefore, the character of the singularity of the exact Hertz function cannot be used for the purpose of obtaining the required condition at $\rho = 0$. For the statement of this condition we have to consider the properties of the Hertz function for large values of kR.

It is known that for large values of kR the so-called "reflection formula" may be used. This formula gives an approximation for the Hertz function in the whole space above the earth's surface, where the inclination of the ray to the horizon is not very small. If the Hertz function is normalized as stated above, the reflection formula may be written

$$U = (1 + f)\,\frac{e^{ikR}}{R}\,, \tag{1.13}$$

where

$$f = \frac{\eta \cos\gamma - \sqrt{\eta - \sin^2\gamma}}{\eta \cos\gamma + \sqrt{\eta - \sin^2\gamma}} \tag{1.14}$$

is the Fresnel coefficient (γ is the incidence angle and $\cos\gamma = z/R$ in our case). The reflection formula is certainly valid in the region where the inequalities

$$1 \ll \frac{kz^2}{2r} \ll k\text{[illegible]} \tag{1.15}$$

are satisfied.

If $|\eta|$ is large and if

$$\frac{1}{\sqrt{|\eta|}} \ll \frac{z}{r} \ll 1 \ , \tag{1.16}$$

then the Fresnel coefficient f is close to unity, and we have

$$U = 2\,\frac{e^{ikR}}{R} \quad . \tag{1.17}$$

When expressed in dimensionless coordinates ρ, ζ, the inequalities (1.15) and (1.16), which are necessary for formula (1.17) to be valid, become

$$1 \ll \frac{\zeta^2}{4\rho} \ll 2|\eta|\rho \ , \tag{1.18}$$

$$1 \ll \frac{\zeta}{\rho} \ll 2\sqrt{|\eta|}. \tag{1.19}$$

To obtain the required condition for W at $\rho \to 0$, we must carry out a double limiting process: firstly $|\eta| \to \infty$ and then $\rho \to 0$. In the limit $|\eta| \to 0$ the right-hand sides of the inequalities may be dropped and we get

$$1 \ll \frac{\zeta^2}{4\rho} ; \quad 1 \ll \frac{\zeta}{\rho} \quad . \tag{1.20}$$

If these relations are satisfied, the Hertz function tends to (1.17) and then

$$W \to 2 \ . \tag{1.21}$$

Inequalities (1.20) are valid particularly for $\rho \to 0$, if $\zeta > 0$. Hence the desired solution of (1.08) has to satisfy the condition

$$|W - 2| \to 0 \ \text{ for } \rho \to 0 \ \text{ and } \zeta > 0 \ . \tag{1.22}$$

However, since $\rho = 0$ is a singular point of the equation for W, condition (1.20) turns out to be not sufficient for the unique determination of the solution. We replace it, therefore, by a more stringent condition

$$\left| \frac{W - 2}{\sqrt{\rho}} \right| \to 0 \quad \text{for } \rho \to 0 \text{ and } \zeta > 0 \,, \tag{1.23}$$

which is, as it will be seen later, a sufficient one.

Thus, for the determination of the "attenuation function" W we have the differential equation (1.08), the boundary conditions (1.10) and (1.23) and the condition of finiteness of U in the region considered (for $\rho > 0$).

To simplify the differential equation, we make the substitution

$$W_1 = \sqrt{\rho}\, e^{-i \frac{\zeta^2}{4\rho}} \quad W_1 \,. \tag{1.24}$$

Then the equation takes the form

$$\frac{\partial^2 W_1}{\partial \zeta^2} + i \frac{\partial W_1}{\partial \rho} = 0 \,. \tag{1.25}$$

The boundary condition for W_1 will be

$$\frac{\partial W_1}{\partial \zeta} + q_1 W_1 = 0 \quad (\text{for } \zeta = 0). \tag{1.26}$$

The condition at $\rho = 0$ becomes

$$\left| W_1 - \frac{2}{\sqrt{\rho}} e^{i \frac{\zeta^2}{4\rho}} \right| \to 0 \quad (\text{for } \rho \to 0). \tag{1.27}$$

Since $\rho = 0$ is a regular point of the equation for W_1 (in distinction to the equation for W) condition (1.27) is a sufficient one.

Solving (1.25) by means of separation of variables, we easily obtain a particular solution which satisfies the boundary condition (1.26); namely

$$W_1 = e^{-i\nu^2\rho} \left(\cos \nu\zeta - \frac{q_1}{\nu} \sin \nu\zeta\right) , \qquad (1.28)$$

where ν is the parameter of separation.

For real values of ν this expression remains finite and satisfies all conditions with the exception of (1.27). For complex values of ν (except the case $\nu = \pm\, iq_1$) expression (1.28) becomes infinite when $\zeta \to \infty$ and therefore, does not satisfy the necessary conditions. If $\nu = \pm\, iq_1$ this expression transforms into the form

$$W_1 = e^{iq_1^2 \rho - q_1 \zeta} . \qquad (1.29)$$

According to (1.11) and (1.12), we have

$$\frac{\pi}{4} < \operatorname{arc} q_1 < \frac{\pi}{2} , \qquad (1.30)$$

and, consequently,

$$\operatorname{Re} (q_1) > 0 ; \quad \operatorname{Re} (iq_1^2) < 0 . \qquad (1.31)$$

Hence the real parts of the coefficients of ρ and ζ in (1.29) are negative and expression (1.28) also satisfies all conditions with the exception of (1.27).

In order to satisfy also the last condition, we construct a function which is a superposition of solutions of the two forms (1.28) and (1.29)

$$W_1 = \int_0^\infty e^{-i\nu^2\rho} \left(\cos \nu\zeta - \frac{q_1}{\nu} \sin \nu\zeta\right) f(\nu)\, d\nu + Ae^{iq_1^2\rho - q_1\zeta} . \qquad (1.32)$$

As easily seen, the singularity of W_1 for $\rho \to 0$ is determined by the behavior of $f(\nu)$ for large values of ν. The required singularity can be represented by the integral

$$\frac{4}{\sqrt{\pi}} e^{i\frac{\pi}{4}} \int_0^\infty e^{-i\nu^2\rho} \cos \nu\zeta \, d\nu = \frac{2}{\sqrt{\rho}} e^{i\frac{\zeta^2}{4\rho}} . \tag{1.33}$$

It is clear, therefore, that at infinity the function $f(\nu)$ tends to a finite limit equal to the constant factor before the integral in (1.33). Let us separate out in (1.32) the term

$$W_1^o = \frac{4}{\sqrt{\pi}} e^{i\frac{\pi}{4}} \int_0^\infty e^{-i\nu^2\rho} \left(\cos \nu\zeta - \frac{q_1}{\nu} \sin \nu\zeta\right) d\nu , \tag{1.34}$$

which corresponds to the limiting value of $f(\nu)$. This term may be transformed into

$$W_1^o = \frac{2}{\sqrt{\rho}} e^{i\frac{\zeta^2}{4\rho}} - q_1 \int_0^\infty \frac{2}{\sqrt{\rho}} e^{i\frac{\zeta^2}{4\rho}} d\zeta . \tag{1.35}$$

W_1^o satisfies equation (1.25) and boundary conditions (1.26). For $\rho \to 0$ we have

$$\lim \left(W_1^o - \frac{2}{\sqrt{\rho}} e^{i\frac{\zeta^2}{4\rho}}\right) = -2\sqrt{\pi}\, e^{i\frac{\pi}{4}} q_1 \tag{1.36}$$

for any $\zeta > 0$. Hence if we put

$$W_1 = W_1^o + W_1' , \tag{1.37}$$

the function W_1' has to satisfy equation (1.25) and condition (1.26), while condition (1.27) gives

$$W_1' = 2\sqrt{\pi}\, e^{i\frac{\pi}{4}} q_1 \quad (\text{for } \rho = 0,\ \zeta > 0) , \tag{1.38}$$

If we put in (1.32)

$$f(\nu) = \frac{4}{\sqrt{\pi}}\, e^{i\frac{\pi}{4}} \Big(1 + g(\nu) \Big) . \tag{1.39}$$

we get

$$W_1' \quad \frac{4}{\sqrt{\pi}}\, e^{i\frac{\pi}{4}} \int_0^\infty e^{-i\nu^2\rho} \left(\cos \nu\zeta - \frac{q_1}{\nu} \sin \nu\zeta \right) g(\nu)\, d\nu + A e^{iq_1^2\rho - q_1\zeta} , \tag{1.40}$$

and condition (1.38) becomes

$$\int_0^\infty \left(\cos \nu\zeta - \frac{q_1}{\nu} \sin \nu\zeta \right) g(\nu)\, d\nu + \frac{\sqrt{\pi}}{4} A e^{-i\frac{\pi}{4}} e^{-q_1\zeta}$$

$$= \frac{\pi}{2} q_1 \quad (\text{for } \zeta > 0) . \tag{1.41}$$

The exponential function in (1.41) admits an integral representation (valid for $\zeta > 0$)

$$e^{-q_1\zeta} = \frac{2}{\pi} q_1 \int_0^\infty \frac{\cos \nu\zeta}{\nu^2 + q_1^2}\, d\nu . \tag{1.42}$$

Multiplying this expression by $q_1 d\zeta$ and integrating over ζ from 0 to ζ we obtain

$$1 - e^{-q_1\zeta} = \frac{2}{\pi} q_1^2 \int_0^\infty \frac{\sin \nu\zeta}{\nu\,(\gamma^2 + q_1^2)}\, d\nu\,. \tag{1.43}$$

Subtracting (1.42) from (1.43) and multiplying by $(\pi/2)q_1$ we get the equation

$$-\, q_1^2 \int_0^\infty \left(\cos \nu\zeta - \frac{q_1}{\nu} \sin \nu\zeta\right) \frac{d\nu}{\nu^2 + q_1^2} + \pi q_1 e^{-q_1\zeta} = \frac{\pi}{2} q_1\,, \tag{1.44}$$

which is to be compared with (1.41). Identifying (1.44) with (1.41) we obtain

$$g(\nu) = -\frac{q_1^2}{\nu^2 + q_1^2}\,; \qquad A = 4\sqrt{\pi}\, e^{i\frac{\pi}{4}} q_1\,. \tag{1.45}$$

According to (1.39), it follows

$$f(\nu) = \frac{4}{\sqrt{\pi}} e^{i\frac{\pi}{4}} \frac{\nu^2}{\nu^2 + q_1^2}\,. \tag{1.46}$$

Inserting this and the value (1.45) for A in (1.32), we arrive at the following expression for the function W_1:

$$W_1 = \frac{4}{\sqrt{\pi}} e^{i\frac{\pi}{4}} \left\{ \int_0^\infty e^{-i\nu^2\rho} \left(\nu \cos \nu\zeta - q_1 \sin \nu\zeta\right) \frac{\nu\, d\nu}{\nu^2 + q_1^2} + \pi q_1 e^{iq_1^2\rho - q_1\zeta} \right\}. \tag{1.47}$$

It is convenient for the investigation of this expression to replace the integral over the real axis by an integral over the line arc $\nu = -\pi/4$, since the new integral converges more rapidly.

In the sector

$$- \frac{\pi}{4} < \arc \nu < 0 \tag{1.48}$$

between the old and the new integration path there is, however, a pole $\nu = - iq_1$. The residue in this pole exactly cancels the additive term in (1.47), and we obtain

$$W_1 = \frac{4}{\sqrt{\pi}} e^{i \frac{\pi}{4}} \int_0^{\infty e^{-i(\pi/4)}} e^{-i\nu^2\rho} (\nu \cos \nu\zeta - q_1 \sin \nu\zeta) \frac{\nu \, d\nu}{\nu^2 + q_1^2} \tag{1.49}$$

We can write instead of this

$$W_1 = \frac{2}{\sqrt{\pi}} e^{i \frac{\pi}{4}} \int_{-\infty e^{-i(\pi/4)}}^{+\infty e^{-i(\pi/4)}} e^{-i\nu^2\rho + i\nu\zeta} \frac{\nu \, d\nu}{\nu - iq_1} \tag{1.50}$$

since the integrand in (1.49) is the even part of the integrand in (1.50). We introduce a new variable of integration p putting

$$\nu = \frac{\zeta}{2\rho} + \frac{p}{\sqrt{\rho}} e^{-i \frac{\pi}{4}} . \tag{1.51}$$

we can shift the contour to the right at the distance $\zeta/2\rho$, then the new variable p will be a real quantity running from $-\infty$ to $+\infty$.

Putting for brevity

$$e^{-i \frac{\pi}{4}} q_1 \sqrt{\rho} = \sigma \; ; \quad e^{i \frac{\pi}{4}} \frac{\zeta}{2\sqrt{\rho}} = \tau , \tag{1.52}$$

we get

$$W_1 = \frac{2}{\sqrt{\pi\rho}}\, e^{i\frac{\zeta^2}{4\rho}} \int_{-\infty}^{+\infty} e^{-p^2}\, \frac{p+\tau}{p+\sigma+\tau}\, dp\,. \qquad (1.53)$$

It is convenient now to go from W_1 back to the original "attenuation function" W, according to (1.24). We shall have

$$W = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-p^2}\, \frac{p+\tau}{p+\sigma+\tau}\, dp\,. \qquad (1.54)$$

This integral can be easily evaluated. It represents different analytic functions according to the sign of the imaginary part of $\sigma + \tau$. But from (1.30) and (1.52) it follows

$$\mathrm{Im}(\sigma) > 0, \quad \mathrm{Im}(\tau) > 0\,, \qquad (1.55)$$

so that in our case $\mathrm{Im}(\sigma + \tau) > 0$. In this case the integral (1.54) is equal to

$$W = 2 - 4\sigma e^{-(\sigma+\tau)^2} \int_{i\infty}^{\sigma+\tau} e^{\alpha^2}\, d\alpha \qquad (1.56)$$

This is the well known Weyl-van der Pol formula, which we have had to derive.

As it is seen from the derivation, the conditions stated above are sufficient to determine the function W in a unique way. On the contrary, any expression of the form (1.32) [with $f(\nu)$ continuous and absolutely integrable] could be added to the obtained solution without interfering with condition (1.22).

As it was already pointed out, the necessity of condition (1.23) is connected with the fact that equation (1.08) for W has a singularity at $\rho = 0$ whereas equation (1.25) for W_1 has no singularities.

The derivation of the Weyl-van der Pol formula by the method of parabolic equation is but little easier than the usual derivation. However, in cases more complicated than the considered case of plane earth the use of this method leads to much greater simplifications.

2. THE CASE OF A SPHERICAL EARTH

Let us denote by r, θ, ϕ spherical coordinates with the origin in the center of the earth globe and with polar axis drawn through the source (vertical dipole). The electric and the magnetic fields can be expressed by means of the Hertz function as follows

$$\left.\begin{aligned} E_r &= \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial U}{\partial\theta}\right), \\ E_\theta &= - \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial U}{\partial\theta}\right); \end{aligned}\right\} \tag{2.01}$$

$$H_\phi = ik \frac{\partial U}{\partial\theta} \quad . \tag{2.02}$$

The function U satisfies the differential equation

$$\Delta U + k^2 U = 0 \tag{2.03}$$

and also certain boundary conditions on the surface of the globe ($r = a$). As in the plane case we shall consider the modulus of the complex inductive capacity η as a large quantity (compared with unity). This assumption permits us to write the boundary conditions in an approximate form pointed out by M. Leontovich, and repeatedly used for the solution of similar problems(2,3). For the plane case these conditions, are of the form (1.09) used by us above; for the spherical case they become

$$\frac{\partial\,(rU)}{\partial r} = - \frac{ik}{\sqrt{\eta}}\; rU \qquad (\text{for } r = a)\ . \tag{2.04}$$

These conditions lead to the following relation for the field components:

$$E_\theta = - \frac{1}{\sqrt{\eta}}\; H_\phi \qquad (\text{for } r = a)\ . \tag{2.05}$$

The character of the singularity of the Hertz function at the point where the dipole is located is the same as in the plane case. Namely, if the dipole and the point of observation are located above the earth's surface and if R is their mutual distance, then it must be

$$\lim RU = 1 \quad \text{for } kR \to 0\ . \tag{2.06}$$

We shall look for the solution of the form

$$U = \frac{e^{ikR}}{R}\; W\ , \tag{2.07}$$

where W is the attenuation function. In the following we shall consider the dipole to be located on the earth's surface itself, and, therefore:

$$R = \sqrt{r^2 + a^2 - 2ra\cos\theta}\ . \tag{2.08}$$

Let us examine what are the "small" and "large" parameters, which characterize our problem. First of all, in the case considered the wave length is extremely small as compared with the radius of the earth. Hence ka is very large, as compared with unity (of the order of several millions).

In solving our problem we shall take this circumstance into account from the very beginning; our aim is to find the asymptotic limiting form of the solution for large values of ka. Further, as pointed out above, we consider $|\eta|$ to be large as compared to unity. The ratio of the orders of magnitude of these two large parameters is to be examined later. At last, we are concerned with distances although large as compared with the wave length, but small as compared with the radius of the earth.

The idea of our method consists in the following. For large ka and large $|\eta|$ the attenuation function W is a slowly varying function of coordinates, i. e. its relative variation over one wave length is very small. This is seen, for instance, from the fact that in a very large region $W = 1 + f$, where f is the Fresnel coefficient, (1.14). To express the slowness of the variation of W in an explicit form we shall introduce large (as compared with the wave length) scales of lengths: m_r in the direction of the radius vector (in the vertical direction) and m_θ in the direction of the meridian arc (in a horizontal direction). Putting

$$r = a + m_r y \; ; \quad \theta = \frac{m_\theta}{a} x \tag{2.09}$$

we introduce new dimensionless quantities x,y and assume that

$$W = W(x,y) \, , \tag{2.10}$$

and that the derivatives $\partial W/\partial x$ and $\partial W/\partial y$ are of the same order of magnitude as W itself (this expresses the slowness of the variation of W). We shall show that by a suitable choice of the scales m_r and m_θ we can (in the case of large ka) obtain for W (x,y) an equation and boundary conditions which do not involve large parameters and which lead to a solution valid in the whole region considered.,

Under our assumptions the equation of the plane of the horizon

$$r \cos \theta = a \tag{2.11}$$

("the boundary of the direct visibility") can be written in the form

$$r = a + a \frac{\theta^2}{2} \tag{2.12}$$

or

$$m_r y = \frac{m_\theta^2}{2a} \quad x^2 \ . \tag{2.13}$$

From considerations of physical nature it is clear that the boundary of direct visibility must play an essential role in our problem. Therefore, it is convenient to make its equation free from any parameters. This can be done by connecting m_r and m_θ by the relation

$$m_r = \frac{m_\theta^2}{2a} \ , \tag{2.14}$$

in virtue of which the equation of the boundary of direct visibility assumes the form

$$y = x^2 \ . \tag{2.15}$$

As mentioned above, we look for the solution in the region where $\theta << \pi/2$. Therefore, we require that to small values of θ should correspond values of x of the order of unity. This will be the case if $m_\theta << a$ or, if we put $m = a/A$, we must consider A as a large number (as compared with unity). Equations (2.01) transform into

$$r = a \left(1 + \frac{1}{2A^2}\right) \ ; \quad \theta = \frac{x}{A} \ , \tag{2.16}$$

and the distance R from the dipole [formula (2.08)], when expressed in terms of x and y, reduces to

$$R = a \frac{x}{A} \left\{ 1 + \frac{1}{4A^2} \left(y + \frac{y^2}{2x^2} - \frac{x^2}{6} \right) \right\} , \tag{2.17}$$

where in the curly brackets the omitted terms are of the order $1/A^4$ and higher.

Let us now derive the approximate differential equation for the attenuation function W. If $\vec{R}$ is the radius vector drawn from the dipole, then from (2.03) and (2.07) follows the equation

$$\Delta W + 2\left(ik - \frac{1}{R}\right)\frac{(\vec{R}\cdot \operatorname{grad} W)}{R} = 0 . \qquad (2.18)$$

Transformed to polar coordinates equation (2.18) takes the form

$$\frac{\partial^2 W}{\partial r^2} + \frac{2}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} + \frac{\operatorname{ctg}\theta}{r}\frac{\partial W}{\partial \theta} +$$

$$+ \frac{2}{R}\left(ik - \frac{1}{R}\right)\left\{(r - a\cos\theta)\frac{\partial W}{\partial r} + \frac{a}{r}\sin\theta\frac{\partial W}{\partial \theta}\right\} = 0 . \qquad (2.19)$$

Making a further transformation from the variables r and θ to x and y and retaining in the differential equation thus obtained only terms of the highest order in A, we get

$$\frac{\partial^2 W}{\partial y^2} + \frac{ika}{2A^3}\left\{\left(x + \frac{y}{x}\right)\frac{\partial W}{\partial y} + \frac{\partial W}{\partial x}\right\} = 0 . \qquad (2.20)$$

We note that the omitted terms are of the order $1/A^2$ compared with those written down.

As yet we have not fixed the value of the large parameter A. We can try to choose its value in such a manner that for $ka \to \infty$ equation (2.20) does not contain any parameters and that is possesses a solution satisfying the necessary conditions. This is only possible if A^3 is proportional to ka. Therefore, we put

$$A = \left(\frac{ka}{2}\right)^{1/3} , \qquad (2.21)$$

and equation (2.20) takes the form

$$\frac{\partial^2 W}{\partial y^2} + i\left[\left(x + \frac{y}{x}\right)\frac{\partial W}{\partial y} + \frac{\partial W}{\partial x}\right] = 0 . \qquad (2.22)$$

We note that this equation is simply the equation for the zero-order term $W^{(0)}$ in the expansion

$$W = W^{(0)} + \frac{1}{A^2} W^{(0)'} + \ldots \qquad (2.23)$$

Besides the assumption that A^3 is proportional to ka, one could consider two more possibilities. Firstly, we could suppose that

$$\frac{A^3}{ka} \to 0 \quad \text{for } ka \to \infty \qquad (2.24)$$

or, secondly, that

$$\frac{A^3}{ka} \to \infty \quad \text{for } ka \to \infty . \qquad (2.25)$$

In the first case the limiting form of the equation would be

$$\left(x + \frac{y}{x}\right) \frac{\partial W}{\partial y} + \frac{\partial W}{\partial x} = 0 , \qquad (2.26)$$

and in the second case:

$$\frac{\partial^2 W}{\partial y^2} = 0 . \qquad (2.27)$$

However, it is easy to prove that the solutions of these equations cannot satisfy the boundary conditions. Thus the only admissible assumption is that made above.

We have now to formulate the boundary conditions. Using (2.17) and (2.21) and retaining only the terms of highest order with respect to A we obtain from (2.04) and (2.07)

$$\frac{\partial \left(e^{ikR} W\right)}{\partial y} = - i \frac{A}{\sqrt{\eta}} \left(e^{ikR} W\right) \quad (\text{for } y = 0) \qquad (2.28)$$

or in the same approximation

$$\frac{\partial W}{\partial y} + i \left(\frac{A}{\sqrt{\eta}} + \frac{x}{2}\right) W = 0 \quad (\text{for } y = 0) . \qquad (2.29)$$

This boundary condition involves the complex quantity

$$q = i\,\frac{A}{\sqrt{\eta}} = i\left(\frac{\pi a}{\lambda}\right)^{1/3} \frac{1}{\sqrt{\epsilon + i\,\frac{\lambda}{2\pi \ell}}}\,, \qquad (2.30)$$

which may be written in the form

$$q = |q| \cdot q_1\,, \qquad (2.31)$$

where the value of q_1 is given by (1.11) and $|q_1| = 1$. Since $|q|$ is the ratio of two large parameters, the value of this quantity can be large as well as small.

We introduce a length b (which is independent of the wave length)

$$b = \left(\frac{a}{2}\right)^{2/5} \ell^{3/5} \qquad (2.32)$$

and put

$$n = \frac{2\pi b}{\lambda}\;;\quad \alpha = \epsilon\,\frac{\ell}{b} = \epsilon\left(\frac{2\ell}{a}\right)^{2/5}. \qquad (2.33)$$

Then the quantity q can be written in the form

$$q = n^{5/6}\,\frac{i}{\sqrt{1 + \alpha n}}\,. \qquad (2.34)$$

As it is seen from Table 1, the parameter a varies for sea water and for different kinds of soil in relatively narrow

TABLE 1

Soil	$\frac{\sigma_o}{\sigma}$	$2\pi\ell$ in meters	$2\pi b$ in meters	ϵ	α
Sea water very salty. . .	2.10^5	0.0032	26.6	80	0.010
Sea water scarcely salty.	10^6	0.016	69.8	80	0.018
Ditto	2.10^6	0.032	105	80	0.024
Swamp	10^7	0.16	278	15	0.009
Moist soil	10^8	1.6	1110	15	0.022
and meadows	2.10^8	3.2	1680	15	0.029
Fresh clean water	10^9	16	4420	80	0.29
Dry soil.	10^{10}	160	17500	9	0.08

Note. The first column gives the ratio of the conductivity of mercury σ_o to the conductivity of a given soil σ. The conductivity of mercury taken to be $\sigma_o = 10440\ (\Omega \cdot \mathrm{cm})^{-1}$.

limits (approximately from 0.01 to 0.03 and for dry soil to 0.08), whereas the length $2\pi b$ varies from tens to thousands of meters. Therefore, n will be very large (such that $|q|$ is of the order of A^2) only for very short waves and dry soils. In the general case, however, we must consider $|q|$ as finite and retain q in the boundary condition which we shall write in the form

$$\frac{\partial W}{\partial y} + \left(q + \frac{ix}{2}\right) W = 0 \quad (\text{for } y = 0) . \tag{2.35}$$

It is interesting to compare the equations and the boundary conditions for the two cases considered (the case of the plane earth and that of the spherical earth). Putting

$$\rho = |q|^2 x, \quad \zeta = |q| y , \tag{2.36}$$

we go back from our variables x,y to the old dimensionless variables ρ, ζ used in §1. Introducing in (2.22) and (2.35) the variables ρ, ζ we obtain the equations

$$\frac{\partial^2 W}{\partial \zeta^2} + i \left[\frac{\partial W}{\partial \rho} + \left(\frac{\zeta}{\rho} + \frac{\rho}{|q|^3} \right) \frac{\partial W}{\partial \zeta} \right] = 0 , \tag{2.37}$$

$$\frac{\partial W}{\partial \zeta} + \left(q_1 + \frac{i\rho}{2|q|^3} \right) W = 0 , \tag{2.38}$$

where the terms of the order $\frac{\rho}{|q|^3}$ are due to the curvature of the earth. By omitting these terms, we return to equations (1.08) and (1.10) for the plane earth.

We have now to formulate the condition at $x = 0$. The corresponding condition for the plane earth has been discussed in §1. It has been shown there that we cannot utilize directly the character of the singularity of the Hertz function in the source, but have to consider the region, where the "reflection formula" (1.13) or its limiting form (1.17) is valid and have to compare these formulas with the desired solution in that region.

For the spherical earth the condition at $x = 0$ does not differ essentially from the corresponding condition for the plane earth, and we can write it in the form

$$\left| \frac{W - 2}{x} \right| \longrightarrow 0 \text{ for } x \longrightarrow 0 \text{ and } y > 0 \tag{2.39}$$

in close analogy to (1.23).

Thus, our problem is to obtain the function W from the differential equation (2.22), conditions (2.35) and (2.39), and the condition that W remains finite for all y > 0.

The solution of this problem, which is of purely mathematical nature, can be obtained as follows.

First of all, we simplify the differential equation (2.22) by the substitution

$$W = e^{-i\omega_o} V , \tag{2.40}$$

where

$$\omega_o = \frac{y^2}{4x} + \frac{xy}{2} - \frac{x^3}{12} . \tag{2.41}$$

The geometrical interpretation of the quantity ω_o follows from formula (2.17) which can be written in the form

$$kR = ka\theta + \omega_o . \tag{2.42}$$

Thus ω_o is the difference between the distance R measured along the straight line and the corresponding length of the arc (measured along the earth's surface), both quantities being expressed in wave numbers. According to (2.40) and (2.42) we have

$$e^{ikR} W = e^{ika\theta} V , \tag{2.43}$$

so that the transition from W to V corresponds to the separation of the phase factor $e^{ika\theta}$ instead of e^{ikR}.

Inserting (2.40) into the differential equation for W and using the relations

$$2\frac{\partial \omega_o}{\partial y} = x + \frac{y}{x} \;;\; \left(\frac{\partial \omega_o}{\partial y}\right)^2 + \frac{\partial \omega_o}{\partial x} = y \;;$$

$$\frac{\partial^2 \omega_o}{\partial y^2} = \frac{1}{2x} \;, \tag{2.44}$$

we obtain

$$\frac{\partial^2 V}{\partial y^2} + i\frac{\partial V}{\partial x} + \left(y - \frac{i}{2x}\right) V = 0 \;. \tag{2.45}$$

This equation (like the original one) has a singularity at $x = 0$, but this singularity can be removed by the substitution

$$V = \sqrt{x}\, W_1 \;. \tag{2.46}$$

The result is

$$\frac{\partial^2 W_1}{\partial y^2} + i\frac{\partial W_1}{\partial x} + yW_1 = 0 \;. \tag{2.47}$$

The boundary condition for W_1 is the same as for V, namely:

$$\frac{\partial W_1}{\partial y} + qW_1 = 0 \quad (\text{for } y = 0) \;. \tag{2.48}$$

We note that this condition is most simply obtained directly from (2.28) [rather than from (2.35)] .

Finally, the condition for $x \to 0$ is

$$W_1 - \frac{2}{\sqrt{x}}\, e^{i\frac{y^2}{4x}} \to 0 \quad (\text{for } x \to 0 \text{ and } y > 0) \;. \tag{2.49}$$

Transition from W to W_1 simplifies the problem considerably. Firstly, equation (2.47) is not only free from a singularity at $x=0$, but also its coefficients do not contain the argument x; therefore, it is soluble by the method of separation of variables. Secondly, the coefficient in the boundary condition (2.48) does not involve x. From the fact that $x=0$ is a regular point of equation (2.47), it follows also that condition (2.49) for $x=0$ (together with the other boundary condition) is sufficient for a unique determination of W_1.

We shall solve equation (2.09) by the classical method of separation of variables. Considering particular solutions of the form

$$W_1 = X(x)\ Y(y) \tag{2.50}$$

we get the following equations for X and Y

$$\frac{Y''}{Y} + y = -i\frac{X'}{X} = t , \tag{2.51}$$

where t is the parameter of separation. Hence

$$X' = it\ X , \tag{2.52}$$

$$Y'' + (y - t)\ Y = 0 . \tag{2.53}$$

The solution of equations (2.52) and (2.53) is

$$X(x) = e^{itx} , \tag{2.54}$$

$$Y(y) = w\ (t - y) , \tag{2.55}$$

where w(t) is an integral of the equation

$$w''(t) = tw(t) . \tag{2.56}$$

For w(t) we may take the function

$$w(t) = \frac{1}{\sqrt{\pi}} \int_{\Gamma} e^{tz - \frac{1}{3} z^3} dz , \qquad (2.57)$$

where the contour Γ is a broken line drawn from infinity to zero along the straight line arc $z = - 2\pi/3$ and from zero to infinity along the positive real axis. The function w(t) is an integral transcendental function which can be expressed through the Hankel function of the first kind and of the order 1/3 according to the formula

$$w(t) = e^{i\frac{2\pi}{3}} \sqrt{\frac{\pi}{3}} \, (-t)^{\frac{1}{2}} \, H^{(1)}_{1/3} \left[\frac{2}{3} (-t)^{2/3} \right] . \qquad (2.58)$$

The properties of w(t) are summarized in I. The function $w(t - y)$ remains finite for $y \rightarrow + \infty$. The second integral of equation (2.53) which may be written in the form

$$Y_2 (y) = w \left[e^{i\frac{2\pi}{3}} (t - y) \right] \qquad (2.59)$$

does not possess this property and must be rejected. Expression (2.50) will satisfy the boundary condition (2.48) if we choose the parameter t so as to satisfy the relation

$$w'(t) - qw(t) = 0 . \qquad (2.60)$$

As it was shown in I all roots t_s of this equation lie in the first quadrant of the t-plane; the distant roots are situated near the straight line arc $t = \pi/3$. Therefrom follows that the function

$$W_1 = e^{ixt_s} w(t_s - y) \tag{2.61}$$

remains finite for all positive values of x and y, satisfies the differential equation (2.47) and the boundary condition (2.48). All these conditions are also satisfied by the function

$$W_1 = \int_C \frac{e^{ixt}\, w(t - y)}{w'(t) - qw(t)} \psi(t)\, dt , \tag{2.62}$$

where C is a closed contour in the t-plane containing the roots of (2.60) and $\psi(t)$ is holomorphic inside this contour.

We have now to satisfy equation (2.49). This can be done by a suitable choice of the contour C and of the function $\psi(t)$. It is clear that the contour C must go to infinity, since the integral along any finite contour cannot have a singularity at $x = 0$. The singularity is caused by distant parts of the contour. But for large values of $|t|$ the following asymptotic expressions are valid

$$\frac{w(t - y)}{w'(t) - qw(t)} \left\{ \begin{array}{ll} \dfrac{1}{\sqrt{t}} e^{-y\sqrt{t}} & \left(-\dfrac{2\pi}{3} \leqslant \text{arc } t < \dfrac{\pi}{3}\right), \\ -\dfrac{1}{\sqrt{t}} e^{y\sqrt{t}} & \left(\dfrac{\pi}{3} < \text{arc } t \leqslant \dfrac{4\pi}{3}\right), \end{array} \right. \tag{2.63}$$

where $\text{arc}\sqrt{t} = \frac{1}{2}\,\text{arc } t$ $\left(\text{on the ray arc } t = \frac{4\pi}{3} \text{ or arc } t = -\frac{2\pi}{3}\right.$ the two expressions coincide$\left.\right)$. The contour C has two branches going to infinity. We shall draw one of them along the positive imaginary axis (from $i\infty$ to 0) and the other along the positive real axis (from 0 to $+\infty$); the lower expression (2.63) is valid on the first branch, the upper - on the second branch. The singularity of the integral (2.62) for $x = 0$ is the same as that of the integral

$$W_1' = \int_0^{i\infty} e^{ixt + y\sqrt{t}}\,\psi(t)\,\frac{dt}{\sqrt{t}} + \int_0^{\infty} e^{ixt - y\sqrt{t}}\,\psi(t)\,\frac{dt}{\sqrt{t}} . \tag{2.64}$$

This is true in spite of the fact that the asymptotic expressions (2.63) are invalid for small and finite values of t, because the integrals over the corresponding parts of the integration path remain finite and have no singularities.

Assuming the function $\psi(t)$ holomorphic and bounded in the first quadrant we can replace the upper limit in the second integral by $i\,\infty$ Then, putting $t = ip^2$, we get

$$W_1' = 2e^{i\frac{\pi}{4}} \int_{-\infty}^{+\infty} e^{-xp^2 + \sqrt{i}\,yp}\,\psi(ip^2)\,dp . \tag{2.65}$$

But we have

$$\int_{-\infty}^{+\infty} e^{-xp^2 + \sqrt{i}\,yp}\,dp = \sqrt{\frac{\pi}{x}}\,e^{i\frac{y^2}{4x}} . \tag{2.66}$$

Therefore, if we suppose that $\psi(t)$ is a constant quantity equal to

$$\psi(t) = \frac{1}{\sqrt{\pi}}\,e^{-i\frac{\pi}{4}} , \tag{2.67}$$

we obtain

$$W_1' = \frac{2}{\sqrt{x}}\,e^{i\frac{y^2}{4x}} , \tag{2.68}$$

which is the required singularity of W_1. Inserting the obtained value of $\psi(t)$ in (2.62) we are led to consider the integral

$$W_1 = \frac{1}{\sqrt{\pi}} e^{-i\frac{\pi}{4}} \int_C \frac{e^{ixt}\, w(t-y)}{w_1'(t) - qw(t)}\, dt \,, \qquad (2.69)$$

which satisfies the differential equation and the boundary conditions and has the required singularity for $x = 0$. However, we cannot yet assert that the integral (2.69) gives the solution of our problem. In fact, the more general form (2.62) of the integral will have the same singularity, if the function $\psi(t)$ is holomorphic in the first quadrant and tends to a constant value (2.67) at infinity. The more general integral satisfies the following relation

$$\lim_{x \to 0} \left(W_1 - \frac{2}{\sqrt{x}} e^{i\frac{y^2}{4x}} \right) = f(y) \,, \qquad (2.70)$$

where $f(y)$ is some bounded function, the form of which depends on $\psi(t)$. But if $\psi(t)$ is a constant, the function $f(y)$ turns out to vanish identically. This can be shown by evaluating the integral (2.69) by the method of steepest descent (the main part of the integration path lies in the neighborhood of the point $t = -\left[(y - x^2)/2x\right]^2$, i. e. for large negative values of t). We shall not perform these calculations since similar ones are made in I.

Hence expression (2.69) satisfies all conditions including (2.49).

We shall not attempt to give here a rigorous proof of the uniqueness of the solution, but it is clear that by adding expressions of the form (2.61) to the solution obtained condition (2.49) is violated.

Going back, according to (2.46), to the function V, we get the following expression for this function:

$$V(x, y, q) = e^{-i\frac{\pi}{4}} \sqrt{\frac{x}{\pi}} \int_C \frac{e^{ixt}\, w(t - y)}{w'(t) - q w(t)}\, dt \quad . \tag{2.71}$$

Using (2.43) and substituting in the denominator of (2.07) $a\theta$ for R, we come to the final expression for the Hertz function

$$U = \frac{e^{ika\theta}}{a\theta} V(x, y, q) \quad . \tag{2.72}$$

This expression coincides exactly with that obtained in I by the method of summation of series.

A detailed discussion of the expression obtained was given in I and shall not be repeated here.

Comparing the two methods of derivation of formula (2.71) we arrive at the following conclusions. The method of the summation of series is more cumbersome but it is at the same time more rigorous. This is connected with the fact that all approximations are made in the ready solution, which makes the estimation of the order of disregarded terms easier. The method permits also to use condition (2.06) directly without resorting to the "reflection formula" which requires a foundation itself. On the other hand, for the method of parabolic equation it is characteristic that all neglections are made in the initial equations. This requires delicate reasoning which is difficult to perform with a complete rigour. The lack of rigour is compensated by the comparative simplicity of the second method. This simplicity is the chief advantage of the method since it gives the possibility to find approximate solutions of other more difficult problems of the same kind where the exact solution is unknown.

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V. THE FIELD OF A PLANE WAVE NEAR THE SURFACE OF A CONDUCTING BODY

V. Fock

For the field induced by an incident plane wave on and near the surface of a convex body of finite conductivity approximate formulas are derived. Since these formulas give also the current distribution in the skin-layer on the surface, they may be used for the calculation (by means of definite integrals) of the field at arbitrary distances from the body, yielding thus an approximate solution of the general diffraction problem.

INTRODUCTION

In our paper "Distribution of Currents Induced by a Plane Wave on the Surface of a Conductor"[1] the following fundamental result has been obtained. The values of the tangential components of the total magnetic field on the surface of a perfect conductor are equal to the surface values of the corresponding components of the field of the incident wave multiplied by a certain universal function $G(\xi)$, depending on the argument $\xi = \ell/d$, where ℓ is the distance from the geometrical boundary of the shadow, measured in the plane of incidence and d is the width of the penumbra region. The quantity d is equal to $d = \sqrt[3]{\frac{\lambda}{\pi}} R_0^2$, where λ is the wave length and R_0 is the curvature radius of the normal section of the surface by the plane of incidence. The surface current density being proportional and directed at right angles to the magnetic field. This result immediately gives the current distribution on the surface, the knowledge of which enables the calculation of the amplitude of the scatter wave.

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p.96

In this paper we intend to generalize this result in two respects.

Firstly, we shall find the field distribution not only on the surface of the body, but also in its neighborhood (at distances that are small as compared with the curvature radii of the surface). Secondly, we shall not consider the body to be a perfect conductor but shall regard it instead as a good conductor only in the sense that on its surface the Leontovich conditions for the tangential field components are valid.

The method we shall use will also differ from that used in the previous paper. In the previous paper we have obtained our result by making use of the local character of the field in the penumbra region. We started from the exact solution of the problem for a particular case and then performed the approximate summation of the series. By the principle of the local field the result could be applied to the general case also. Now we shall find the solution directly for the general case of an arbitrary surface, using the method of parabolic equation proposed by Leontovich and developed in our common paper[2] for the case of a point source (dipole), located on a plane or on a spherical surface.

1. THE GEOMETRICAL ASPECT OF THE PROBLEM

Consider a convex body and a plane wave incident in the direction of the x axis. If the equation of the surface of the body is

$$f\ (x,y,z) = 0\ , \tag{1.01}$$

then the equation of the curve, representing the boundary of the geometrical shadow on the surface, will be obtained from the equation of the surface and the relation

$$\frac{\partial f}{\partial x} = 0\ . \tag{1.02}$$

Let us take on the surface a point lying on the boundary of the geometrical shadow and consider it to be the origin of our coordinate system. The z axis we direct along the normal to the surface (towards the air). Since on the shadow boundary the normal is perpendicular to the direction of the wave, the z axis so chosen will be perpendicular to our x axis. The direction of the y axis we choose in such a way as to obtain a right-handed coordinate system.

In the vicinity of any given point the equation of the surface will be of the form

$$z + \tfrac{1}{2}\,(ax^2 + 2bxy + cy^2) = 0 \; . \qquad (1.03)$$

Since the surface is convex and the z axis is directed to the convex side we have

$$a > 0 \; ; \quad c > 0 \; ; \quad ac - b^2 \gg 0 \; . \qquad (1.04)$$

The equation of the cylindrical surface which separates the region of the geometrical shadow is obtained by eliminating x from (1.01) and (1.02). In our case this equation will be of the form

$$z + \frac{ac - b^2}{2a}\, y^2 = 0 \; . \qquad (1.05)$$

The curvature radius of the normal section of the surface by the plane of incidence is equal to

$$R_o = \frac{1}{a} \; . \qquad (1.06)$$

Our problem is to find the electromagnetic field near the surface, at distances (from the surface and from the origin) that are small as compared to the curvature radius R_o.

2. SIMPLIFIED MAXWELL'S EQUATIONS

We suppose the time dependence of the field components to be of the form $e^{-i\omega t}$ and omit this factor in the following. By k we denote the absolute value of the wave vector

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} . \tag{2.01}$$

Each of the field components satisfies Helmholtz's equation

$$\Delta\Psi + k^2\Psi = 0 . \tag{2.02}$$

where Δ is the Laplace operator. Since we deal with a field, due to a plane wave traveling in the direction of the x axis, we shall separate out the factor e^{ikx} in Ψ and put

$$\Psi = e^{ikx}\,\Psi^* . \tag{2.03}$$

Then Ψ^* will satisfy the equation

$$\frac{\partial^2\Psi^*}{\partial x^2} + \frac{\partial^2\Psi^*}{\partial y^2} + \frac{\partial^2\Psi^*}{\partial z^2} + 2ik\frac{\partial\Psi^*}{\partial x} = 0 . \tag{2.04}$$

The field components satisfy the Maxwell equations

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = ik\,H_x , \text{ etc.,} \tag{2.05}$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = -\,ik\,E_x , \text{ etc.} \tag{2.06}$$

Let us now separate out in each of the field components the factor e^{ikx} and put

$$E_x = E_x^*\;e^{ikx}, \text{ etc; } \quad H_x = H_x^*\;e^{ikx} , \text{ etc.} \tag{2.07}$$

In this way we obtain for the quantities marked by an asterisk the equations:

$$\left.\begin{aligned}
&\frac{\partial E_z^*}{\partial y} - \frac{\partial E_y^*}{\partial z} = ik\, H_x^* \,, \\
&\frac{\partial E_x^*}{\partial z} - \frac{\partial E_z^*}{\partial x} - ik\, E_z^* = ik\, H_y^* \,, \\
&\frac{\partial E_y^*}{\partial x} - \frac{\partial E_x^*}{\partial y} + ik\, E_y^* = ik\, H_z^* \,;
\end{aligned}\right\} \qquad (2.08)$$

$$\left.\begin{aligned}
&\frac{\partial H_z^*}{\partial y} - \frac{\partial H_y^*}{\partial z} = -\, ik\, E_x^* \,, \\
&\frac{\partial H_x^*}{\partial z} - \frac{\partial H_z^*}{\partial x} - ik\, H_z^* = -\, ik\, E^* \,, \\
&\frac{\partial H_y^*}{\partial x} - \frac{\partial H_x^*}{\partial y} + ik\, H_y^* = -\, ik\, E_z^* \,.
\end{aligned}\right\} \qquad (2.09)$$

We shall now introduce an assumption which will be of primary importance for the following; namely, we suppose that the quantities with asterisks are slowly varying functions of coordinates in the sense that their relative variation along the distance of one wave length is small.

Besides, we suppose that the variation of these quantities in the z direction (normal to the surface) takes place more rapidly than in the x and y directions (parallel to the surface).

These assumptions can be stated in the form

$$\frac{\partial \Psi^*}{\partial z} = 0\left(\frac{k}{m}\ \Psi^*\right) ; \quad \frac{\partial \Psi^*}{\partial x} = 0\left(\frac{k}{m'}\ \Psi^*\right) ; \quad \frac{\partial \Psi^*}{\partial y} = 0\left(\frac{k}{m'}\ \Psi^*\right) \tag{2.10}$$

m and m' being dimensionless parameters and

$$m' \gg m \gg 1 \ . \tag{2.11}$$

The truth of these assumptions follows from the fact that the final solution (which is unique) actually satisfies them.

It follows from these assumptions that the second derivatives with respect to x and y in equation (2.04) are small as compared to the second derivative with respect to z. Hence this equation takes the form

$$\frac{\partial^2 \Psi^*}{\partial z^2} + 2\ ik\ \frac{\partial \Psi^*}{\partial x} = 0 \ . \tag{2.12}$$

It follows from (2.12) that m' is of the order of m^2 and we can put

$$m' = m^2 \ . \tag{2.13}$$

The relations (2.10) can now be written in the form

$$\frac{\partial \Psi^*}{\partial x} = 0\left(\frac{k}{m^2}\ \Psi^*\right) ; \quad \frac{\partial \Psi^*}{\partial y} = 0\left(\frac{k}{m^2}\ \Psi^*\right) ; \quad \frac{\partial \Psi^*}{\partial z} = 0\left(\frac{k}{m}\ \Psi^*\right) \ . \tag{2.14}$$

From relations (2.14) (that are valid for all the field components) it follows that in equation (2.12) the terms omitted are of the order $1/m^2$ as compared with those written down. Terms of this order of magnitude shall always be neglected in the following.

Let us estimate on the basis of (2.14) the order of magnitude of the different terms in equations (2.08) and (2.09). In doing this, we consider H_y^* and H_z^*, as the principal quantities to which all the other quantities are to be compared. As to the relative order of magnitude of H_y^* and H_z^*, we shall suppose the order of one of these quantities to differ from that of the order, at the most, by the factor m.

From the first equation (2.09) we get

$$E_x^* = 0\left(\frac{1}{m} H_y^*\right) + 0\left(\frac{1}{m^2} H_z^*\right) . \tag{2.15}$$

Inserting this estimation into the second equation (2.08) we see that the term $\partial E_x^*/\partial z$ is very small (of the order of $1/m^2$) as compared to the term ik H_y^*. On the other hand, it is seen directly from (2.14) that the term $\partial E_z^*/\partial x$ is of the order $1/m^2$ as compared with ik E_z^*. The term of this order of magnitude must be disregarded. Then the second equation (2.08) gives simply $E_z^* = H_y^*$. Similarly the third equation (2.08) gives $E_y^* = H_z^*$, and the first equation (2.08) shows that H_x^* will be of the order

$$H_x^* = 0\left(\frac{1}{m^2} H_y^*\right) + 0\left(\frac{1}{m} H_z^*\right) . \tag{2.16}$$

These values are also in agreement with equations (2.09).

Hence all the field components may be expressed, with neglect of small quantities, in terms of H_y^* and H_z^*. Since these expressions do not involve derivatives with respect to x, they have the same form for the field components without an asterisk, namely:

$$\left.\begin{aligned}
E_x &= \frac{i}{k}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) , \\
E_y &= H_z , \\
E_z &= - H_y , \\
H_x &= \frac{i}{k} \quad \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} \quad .
\end{aligned}\right\} \tag{2.17}$$

The last equation can be obtained also directly from div H = 0. To these equations we must add the Helmholtz equation for each of the field components or the equation of the form (2.12) for the quantities with asterisks.

3. SIMPLIFIED BOUNDARY CONDITIONS

As shown by Leontovich, if the absolute value of the complex inductive capacity of the medium

$$\eta = \epsilon + i\,\frac{4\pi\sigma}{ck} \tag{3.01}$$

is great as compared to unity, there is no need to consider the field within the medium, but one may take into account the influence of the medium on the field in the air by means of the boundary conditions, connecting the tangential components of this field on the surface of the reflecting body.

Leontovich's conditions (to be more correct, their generalization to the case when the magnetic permeability of the medium is different from unity) can be written in the form of three equations:

$$\left.\begin{aligned} E_x - n_x E_n &= \sqrt{\frac{\mu}{\eta}}\left(n_y H_z - n_z H_y\right) , \\ E_y - n_y E_n &= \sqrt{\frac{\mu}{\eta}}\left(n_z H_x - n_x H_z\right) , \\ E_z - n_z E_n &= \sqrt{\frac{\mu}{\eta}}\left(n_x H_y - n_y H_x\right) , \end{aligned}\right\} \qquad (3.02)$$

only two of which are independent.

In these equations n_x, n_y, n_z are components of the unit vector of the normal to the surface and E_n has the value

$$E_n = n_x E_x + n_y E_y + n_z E_z . \qquad (3.03)$$

It can be shown that the conditions (3.02) are valid if the following inequalities are satisfied

$$|\eta\mu| \gg 1 , \qquad (3.04)$$

$$kR_o \sqrt{|\eta\mu|} \gg 1 , \qquad (3.05)$$

where R_o is the smallest curvature radius of the normal section of the surface.

In the case of a conductor, in which the displacement current is negligible, these inequalities have the following meaning. According to the first inequality the square of the depth of the skin-effect layer must be small as compared to the square of the wave length in air. According to the second inequality this depth must be small as compared with the curvature radius of the normal section of the surface.

In the following we put the magnetic permeability equal to unity and transform conditions (3.02) using the relations $E_y = H_z$ and $E_z = - H_y$ obtained above. From (3.02) we get

$$\left(1 - n_x^2\right) E_x = \left(n_x + \frac{1}{\sqrt{\eta}}\right) \left(n_y H_z - n_z H_y\right) , \quad (3.06)$$

$$\left(1 - n_x^2\right) H_x = \left(n_x + \frac{1}{\sqrt{\eta}}\right) \left(n_y H_y + n_z H_z\right) . \quad (3.07)$$

Using for H_x the estimate (2.16) and considering the quantity $\sqrt{\eta}$ to be large (of the order of m or larger), we infer that the left-hand side of (3.07) is small as compared with the separate terms of the right-hand side. Replacing this quantity by zero we obtain instead of (3.07)

$$n_y H_y + n_z H_z = 0 . \quad (3.08)$$

Using this relation we get from (3.06)

$$n_z E_x = - \left(n_x + \frac{1}{\sqrt{\eta}}\right) H_y . \quad (3.09)$$

We may insert in this relation the expression for E_x from the first equation (2.17). Since the y axis has a tangential (or an almost tangential) direction, we can differentiate (3.08) with respect to y and put

$$n_y \frac{\partial H_y}{\partial y} + n_z \frac{\partial H_z}{\partial y} = 0 . \quad (3.10)$$

We have omitted in equation (3.10) small terms, depending upon the surface curvature and similar to those which have been neglected when obtaining the condition (3.02). As a result we obtain from (3.09)

$$n_y \frac{\partial H_y}{\partial y} + n_z \frac{\partial H_y}{\partial z} = - ik \left(n_x + \frac{1}{\sqrt{\eta}}\right) H_y . \quad (3.11)$$

This boundary condition contains only one component H_y. Having in mind the dependence of H_y upon x, we can write with the same accuracy

$$\frac{\partial H_y}{\partial n} = - i \frac{k}{\sqrt{\eta}} H_y , \qquad (3.12)$$

where in the left-hand side the derivative is taken along the normal. Now, estimating the order of magnitude of the first term in the left-hand side of (3.11) and considering n_x and n_y to be small of the order of 1/m, we infer that this first term is small as compared with the second one. We shall write, therefore,

$$\frac{\partial H_y}{\partial z} = - ik \left(n_x + \frac{1}{\sqrt{\eta}} \right) H_y . \qquad (3.13)$$

In addition to the differential equation and the boundary condition on the surface of the body, the quantity H_y must satisfy the following requirement (condition at infinity). In the illuminated region at large distances from the shadow boundary the part of H_y, which has the phase of the incident wave, must have a prescribed amplitude. (Under large distances we mean the distances which are still small in comparison with the surface curvature radii although they involve many wave lengths.)

Thus, the field component H_y (and, therefore, E) has been completely separated from the other field components: it satisfies a separate differential equation, a separate boundary condition on the surface of the body and a separate condition at infinity. These conditions determine H_y in a unique way.

After having determined H_y, we can find H_z from the differential equation, condition (3.08) on the surface of the body and from the condition at infinity. The latter condition consists in the requirement that the part of H_z, which corresponds to the incident wave, must have a given amplitude. Finally, knowing H_y and H_z, we can determine all the other field components from equations (2.17).

4. DETERMINATION OF THE FIELD COMPONENT H_y

Let us put

$$H_y = H_y^o e^{ikx} \Psi^* , \tag{4.01}$$

where H_y^o is the amplitude of the incident wave at infinity. According to (2.12) and (3.13) the function Ψ^* must satisfy the equation

$$\frac{\partial^2 \Psi^*}{\partial z^2} + 2ik \frac{\partial \Psi^*}{\partial x} = 0 \tag{4.02}$$

and the boundary condition

$$\frac{\partial \Psi^*}{\partial z} + ik \left(ax + by + \frac{1}{\sqrt{\eta}} \right) \Psi^* = 0 \tag{4.03}$$

on the surface

$$z + \frac{1}{2} \left(ax^2 + 2bxy + cy^2 \right) = 0 . \tag{4.04}$$

We have replaced n_x in (4.03) by its approximate value obtained from the equation of the surface.

Suppose that the function Ψ^* depends upon the coordinates x, y, z only by means of two variables

$$\xi = m (ax + by) , \tag{4.05}$$

$$\zeta = 2am^2 \left[z + \frac{1}{2} \left(ax^2 + 2bxy + cy^2 \right) \right] , \tag{4.06}$$

where m is a large parameter which will be defined below. The scales of the quantities ξ and ζ are chosen in such a way that equation (1.05) (giving the shadow boundary in space) takes the form

$$\zeta = \xi^2 . \tag{4.07}$$

The values of the variable ζ can be only non-negative and those of the variable ξ can be both positive and negative.

In the illuminated region of the space we have $\xi < \sqrt{\zeta}$ and in the shaded one $\xi > \sqrt{\zeta}$, where the square root is taken with a positive sign.

Calculating the derivatives we obtain:

$$\frac{\partial \Psi^*}{\partial y} = ma\left(\frac{\partial \Psi^*}{\partial \xi} + 2\xi \frac{\partial \Psi^*}{\partial \zeta}\right) , \tag{4.08}$$

$$\frac{\partial \Psi^*}{\partial z} = 2m^2 a \frac{\partial \Psi^*}{\partial \zeta} , \tag{4.09}$$

and equation (4.02) takes the form

$$\frac{\partial^2 \Psi^*}{\partial \zeta^2} + i \frac{k}{2m^3 a}\left(\frac{\partial \Psi^*}{\partial \xi} + 2\xi \frac{\partial \Psi^*}{\partial \zeta}\right) = 0 . \tag{4.10}$$

We now choose the parameter m in such a way as to make the coefficient in this equation equal to unity

$$m = \sqrt[3]{\frac{k}{2a}} = \sqrt{\frac{kR_0}{2}} . \tag{4.11}$$

Since we consider the wave length to be very small as compared with the curvature radius of the surface, the value of our parameter m will actually be large. The expressions for the derivatives can now be written in the following form:

$$\frac{\partial \Psi^*}{\partial x} = \frac{k}{2m^2} \left(\frac{\partial \Psi^*}{\partial \xi} + 2\xi \frac{\partial \Psi^*}{\partial \zeta} \right) , \qquad (4.12)$$

$$\frac{\partial \Psi^*}{\partial z} = \frac{k}{m} \frac{\partial \Psi^*}{\partial \zeta} . \qquad (4.13)$$

It is seen from these equations that the estimates (2.14) will be valid, provided the derivatives of Ψ^* with respect to ξ and to ζ are of the order of Ψ^* itself.

Equation (4.10) takes the form

$$\frac{\partial^2 \Psi^*}{\partial \zeta^2} + i \left(\frac{\partial \Psi^*}{\partial \xi} + 2\xi \frac{\partial \Psi^*}{\partial \zeta} \right) = 0 . \qquad (4.14)$$

The boundary condition (4.03) becomes

$$\frac{\partial \Psi^*}{\partial \zeta} + i\xi \, \Psi^* + q \, \Psi^* = 0 , \qquad (4.15)$$

where we have put for brevity:

$$q = \frac{im}{\sqrt{\eta}} = \frac{i}{\sqrt{\eta}} \sqrt[3]{\frac{k}{2a}} . \qquad (4.16)$$

The quantity q will be, in general, finite, but can be also small (for a very good conductor) or large (for an almost plane surface).

The condition at infinity for Ψ^* consists in the following. In the illuminated region that part of Ψ^*, the phase of which vanishes, must have an amplitude equal to unity.

To simplify the differential equation we put

$$\Psi^* = e^{-i\xi\zeta + i(\xi^3/3)} V . \qquad (4.17)$$

Then the equation and the boundary condition for V will be

$$\frac{\partial^2 V}{\partial \zeta^2} + i\,\frac{\partial V}{\partial \xi} + \zeta V = 0\ , \tag{4.18}$$

$$\frac{\partial V}{\partial \zeta} + qV = 0, \quad \zeta = 0\ . \tag{4.19}$$

The condition at infinity (large negative values of ξ) becomes

$$V = e^{i\xi\zeta - i(\xi^3/3)} - V^*\ , \tag{4.20}$$

Where V^* corresponds to the reflected wave. We denote by ϕ the phase of the first term in (4.20)

$$\phi = \xi\zeta - \frac{\xi^3}{3}\ , \tag{4.21}$$

and by ϕ^* the phase of V^*. The phase ϕ^* can be determined by calculating from geometrical considerations the phase difference $\phi^* - \phi$ between the reflected and the incident wave and by using the known value (4.21) of the phase ϕ.

It can be shown that the phase ϕ^* so determined is equal to the extremum value of the function

$$\phi^* = t\xi + \frac{2}{3}(\zeta - t)^{3/2} - \frac{4}{3}(-t)^{3/2}\ , \tag{4.22}$$

i.e. equal to the value of t, for which $\partial\phi^*/\partial t = 0$. Similarly the given phase (4.21) is equal to the extremum value of the function

$$\phi = t\xi - \frac{2}{3}(\zeta - t)^{3/2}\ . \tag{4.23}$$

We omit the derivation, since it is rather cumbersome and since the result can be obtained in a purely analytical way from the final form of the solution (see § 6).

Equation (4.18) coincides with that which occurs in the problem of diffraction of radio waves around the earth's surface. This equation (with different conditions at infinity) was investigated in our previous paper.[2]

Equation (4.18) admits particular solutions of the form

$$V = e^{i\xi t}\, w(t - \zeta) \,, \tag{4.24}$$

where w(t) is an integral of the ordinary differential equation of the second order

$$w''(t) = tw(t) \,. \tag{4.25}$$

We shall need both integrals of equation (4.25). As one of these integrals we take the function

$$w_1(t) = \frac{1}{\sqrt{\pi}} \int_{\Gamma_1} e^{zt - \frac{1}{3} z^3}\, dz \,, \tag{4.26}$$

where the contour Γ_1 goes from infinity to the origin along the ray arc $z = -\frac{2}{3}\pi$ and then returns to infinity along the ray arc $z = 0$ (along the positive real axis). Another (linearly independent) integral is the function

$$w_2(t) = \frac{1}{\sqrt{\pi}} \int_{\Gamma_2} e^{zt - \frac{1}{3} z^3}\, dz \,, \tag{4.27}$$

where the contour Γ_2 is an image of the contour Γ_1 in the real axis of the z plane. For real values of t the functions $w_1(t)$ and $w_2(t)$ are complex-conjugates. We shall have

$$\left.\begin{aligned} w_1(t) &= u(t) + iv(t) \,, \\ w_2(t) &= u(t) - iv(t) \,. \end{aligned}\right\} \tag{4.28}$$

for real functions u(t) and v(t) and their derivatives extensive four-figure tables (range from $t = -9.00$ to $t = +9.00$, interval 0.02) have been calculated by us.[4]

The asymptotic expression for $w_1(t)$, valid for large negative values of t (and also in a certain sector in the plane of the complex variable t) has the form

$$w_1(t) = (-t)^{-1/4} \exp\left(i\frac{2}{3}(-t)^{3/2} + i\frac{\pi}{4}\right). \qquad (4.29)$$

Similarly

$$w_2(t) = (-t)^{-1/4} \exp\left(-i\frac{2}{3}(-t)^{3/2} - i\frac{\pi}{4}\right). \qquad (4.30)$$

From (4.23) and (4.30) we see that the phase of the expression

$$e^{i\xi t}\, w_2(t - \zeta) \qquad (4.31)$$

is just equal to ϕ; and we know that the extremum of ϕ gives the phase of the incident wave. Therefore, we can expect that the integration of the function (4.31) along a contour which passes near the point of the extremum of the phase, gives an expression, the phase of which is equal to that of the incident wave (4.21). In fact, making use of the relations:

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{xp}\, v(p)\, dp = e^{\frac{1}{3}x^3} \quad (\mathrm{Re}\, x > 0), \qquad (4.32)$$

$$w_2\left(pe^{i\frac{2\pi}{3}}\right) = 2e^{-i\frac{\pi}{6}}\, v(p), \qquad (4.33)$$

the following equality may be proved

$$e^{i\xi\zeta - i\frac{\xi^3}{3}} = \frac{1}{2\sqrt{\pi}} \int_C e^{it\xi}\, w_2(t - \zeta)\, dt, \qquad (4.34)$$

where the contour C is described along the ray arc $z = \frac{2}{3}\pi$ from infinity to the origin and along the ray arc $z = -\frac{1}{3}\pi$ from the origin to infinity.

On the other hand, if the function $f(t)$ is such that its phase for large negative values of t is equal to $-\frac{4}{3}(-t)^{3/2}$ then the phase of the expression

$$e^{i\xi t}\, w_1(t-\zeta)\, f(t) \tag{4.35}$$

is equal to ϕ^* [in formula (4.22)] . Hence, integrating expression (4.35) along a contour, which passes in the vicinity of the point of the extremum of the phase, we obtain an expression which has a phase equal to that of the incident wave.

From these considerations it follows that we may seek the expression for V in the form

$$V = \frac{1}{2\sqrt{\pi}} \int_C e^{i\xi t} \left\{ w_2(t-\zeta) - f(t)\, w_1(t-\zeta) \right\} dt\ . \tag{4.36}$$

This expression satisfies equation (4.18) and the condition at infinity (4.20). To satisfy also the boundary conditions (4.19) we have to determine the function $f(t)$ from the relation

$$w_2'(t) - q w_2(t) = f(t) \left\{ w_1'(t) - q w_1(t) \right\} , \tag{4.37}$$

whence

$$f(t) = \frac{w_2'(t) - q w_2(t)}{w_1'(t) - q w_1(t)}\ . \tag{4.38}$$

It is not difficult to see from (4.29) and (4.30) that the obtained function $f(t)$ has the correct phase.

We, finally, obtain

$$V = \frac{1}{2\sqrt{\pi}} \int_C e^{i\xi t} \left\{ w_2(t-\zeta) - \frac{w_2'(t) - q w_2(t)}{w_1'(t) - q w_1(t)} \, w_1(t-\zeta) \right\} dt \, . \tag{4.39}$$

With this value of V the expression

$$H_y = H_y^o \, e^{ikx} \, e^{-i\xi\zeta + i(\xi^3/3)} \, V \tag{4.40}$$

gives the y component of the magnetic field.

Using the relation

$$w_1'(t) \, w_2(t) - w_2'(t) \, w_1(t) = -2i \tag{4.41}$$

it is easy to verify that at $\zeta = 0$ (on the surface of the body) the expression (4.39) for V becomes

$$V = \frac{1}{\sqrt{\pi}} \int_C e^{i\xi t} \frac{dt}{w_1'(t) - q w_1(t)} \, . \tag{4.42}$$

Inserting this in (4.40), we arrive at the following conclusion. The tangential components H_{tg} of the magnetic field on the surface of the body are equal to their values H_{tg}^{ex} for the external field, multiplied by a certain universal function of the reduced distance ξ from the shadow boundary and of the parameter q (the latter depends upon the wave length and the properties of the body). We have

$$H_{tg} = H_{tg}^{ex} \, G(\xi, q) \, , \tag{4.43}$$

where

$$G(\xi,q) = e^{i\xi^3/3} \frac{1}{\sqrt{\pi}} \int_C e^{i\xi t} \frac{dt}{w_1'(t) - qw_1(t)} . \quad (4.44)$$

This result is in agreement with that obtained in our previous paper[1] by a wholly different method and represents its generalization to the case of a finite electrical conductivity of the body.

For a perfect conductor $q = 0$ we have

$$G(\xi,0) = G(\xi) , \quad (4.45)$$

where $G(\xi)$ is a function tabulated in our previous paper.[1]

We note that the quantity V determined by (4.42) occurred also in our solution[3] of the problem of the propagation of radio waves around the earth's surface [it was denoted there by $V_1(\xi,q)$].

5. DETERMINATION OF THE COMPONENT H_z AND THE OTHER FIELD COMPONENTS

We have still to determine the component of the magnetic field H_z with help of the conditions formulated at the end of § 3.

We begin with a particular case, when the magnetic vector is polarized parallel to the z axis. Then $H_y^o = 0$ and, according to the results of § 4, we have in our approximation $H_y = 0$ in all the region considered. Then, according to the boundary condition (3.08), we shall have $H_z = 0$ on the surface of the body.

Let us put

$$H_z = H_z^o e^{ik\cdot} \Phi^* , \qquad (5.01)$$

where H_z^o is the amplitude of the incident wave at infinity.

The function Φ^* must satisfy the equation

$$\frac{\partial^2 \Phi^*}{\partial z^2} + 2ik \frac{\partial \Phi^*}{\partial x} = 0 \qquad (5.02)$$

and the boundary condition

$$\Phi^* = 0 \quad \text{on the surface of the body.} \qquad (5.03)$$

The condition at infinity will be the same as the condition for Ψ^*.

We assume that Φ^* depends on the same variables ξ, ζ as Ψ^* and make the substitution

$$\Phi^* = e^{i\xi\zeta + i(\xi^3/3)} U . \qquad (5.04)$$

Since Φ^* satisfies the same equation as Ψ^*, the equation for U coincides with equation (4.18) for V. For the determination of U we obtain, therefore, the equation

$$\frac{\partial^2 U}{\partial \zeta^2} + i \frac{\partial U}{\partial \xi} + \zeta U = 0 , \qquad (5.05)$$

the boundary condition

$$U = 0 \quad \text{for } \zeta = 0 , \qquad (5.06)$$

and the condition at infinity

$$U = e^{i\xi\zeta - i(\xi^3/3)} U^* , \qquad (5.07)$$

where U^* corresponds to the reflected wave.

If we assume for U an expression of the form (4.36), the function f(t) therein will be determined from equation (5.06) and we obtain

$$U = \frac{1}{2\sqrt{\pi}} \int_C e^{i\xi t} \left\{ w_2(t-\zeta) - \frac{w_2(t)}{w_1(t)} \; w_1(t-\zeta) \right\} dt \; . \qquad (5.08)$$

Inserting (5.08) and (5.04) into (5.01) we obtain the solution of our problem for the particular case $H_y^o = 0$.

Consider now the general case. The boundary condition on the surface has the form

$$H_z = - (bx + cy) \, H_y \; , \qquad (5.09)$$

where H_y is known. Using the identity

$$bx + cy = \frac{b}{a} (ax + by) + \frac{ac - b^2}{a} \, y \; , \qquad (5.10)$$

we can write instead of (5.09)

$$H_z = - \frac{b}{a} (ax + by) \, H_y - \frac{ac - b^2}{a} \, y \, H_y \; . \qquad (5.11)$$

But, in virtue of the boundary condition (3.13), for H_y we have on the surface

$$(ax + by) \, H_y = \frac{i}{k} \frac{\partial H_y}{\partial z} - \frac{1}{\sqrt{\eta}} \, H_y \; . \qquad (5.12)$$

Inserting this value into (5.11) we get

$$H_z = - \frac{b}{a} \left(\frac{i}{k} \frac{\partial H_y}{\partial z} - \frac{1}{\sqrt{\eta}} \, H_y \right) - \frac{ac - b^2}{a} \, y \, H_y \; . \qquad (5.13)$$

This equality is certainly valid on the surface of the body. But owing to the fact that the derivatives with respect to y are not involved in equation (4.02), the right-hand side of (5.13) [unlike that of (5.11)] satisfies also the approximate wave equation in space. Therefore, the value of H_z in space can differ from the value of the right-hand side (5.13) only by a quantity which is a solution of the approximate wave equation and which vanishes on the surface. But a quantity having all these properties is either the function $e^{ikx} \cdot \Phi$ or any function proportional to it (where the proportionality factor can depend on y).

The above considerations permit us to determine the complete expression for H_z in a simple way. We rewrite equation (5.13) inserting for H_y the expression (4.01). We get

$$H_z = -\frac{1}{m} H_y^o e^{ikx} \left\{ i \frac{b}{a} \left[\frac{\partial \Psi^*}{\partial \zeta} + q \Psi^* \right] + \frac{ac - b^2}{a} my \Psi^* \right\} . \tag{5.14}$$

If we add to the right-hand side of (5.14) terms proportional to $e^{ikx} \Phi^*$ and vanishing on the surface, we can also write

$$H_z = -\frac{1}{m} H_y^o e^{ikx} \left\{ i \frac{b}{a} \left[\frac{\partial \Psi^*}{\partial \zeta} + q (\Psi^* - \Phi^*) \right] + \frac{ac - b^2}{a} my (\Psi^* - \Phi^*) \right\} + H_z^o e^{ikx} \Phi^* . \tag{5.15}$$

We shall now show that this expression is valid not only on the surface but also in space (within the whole region considered). It is obvious that it satisfies the approximate wave equation and the boundary conditions. It remains only to show that it satisfies also the condition at infinity. This becomes evident if we note that in the derivative $\partial\Psi^*/\partial\zeta$ and also in the difference $\Psi^* - \Phi^*$ the amplitude of the term corresponding to the incident wave vanishes. Hence at infinity only the term proportional to H_z^o will correspond to the incident wave, and this term has a correct amplitude.

We have obtained the components H_y and H_z. The remaining components can be determined from the simplified Maxwell equations (2.17). Omitting small terms we obtain

$$E_x = -\frac{i}{m} H_y^o e^{ikx} \frac{\partial\Psi^*}{\partial\zeta}, \qquad (5.16)$$

$$H_x = \frac{i}{m} H_z^o e^{ikx} \frac{\partial\Phi^*}{\partial\zeta}. \qquad (5.17)$$

The determination of the field components is now complete.

6. THE FIELD IN THE ILLUMINATED REGION

In order to investigate the field in the illuminated region we have to deduce for the functions U and V given by (5.08) and (4.39) asymptotic expressions, valid for large negative values of ξ.

We put according to (4.21)

$$\phi = \xi\zeta - \frac{1}{3}\xi^3. \qquad (6.01)$$

Then we have

$$U = e^{i\phi} - U^*, \qquad (6.02)$$

$$V = e^{i\phi} - V^*, \qquad (6.03)$$

where

$$U^* = \frac{1}{2\sqrt{\pi}} \int_C e^{i\xi t} \frac{w_2(t)}{w_1(t)} w_1(t - \zeta)\, dt , \qquad (6.04)$$

$$V^* = \frac{1}{2\sqrt{\pi}} \int_C e^{i\xi t} \frac{w_2'(t) - qw_2(t)}{w_1'(t) - qw_1(t)} w_1(t - \zeta)\, dt . \qquad (6.05)$$

The phase of the integrands in U^* and V^* is equal to the expression

$$\phi^* = t\xi + \frac{2}{3}(\zeta - t)^{3/2} - \frac{4}{3}(-t)^{3/2} , \qquad (6.06)$$

which was considered above [formula (4.22)] .

In the point of the extremum of the phase we have

$$\sqrt{-t} = \frac{1}{3}\sigma - \frac{2}{3}\xi , \qquad (6.07)$$

$$\sqrt{-t} = \frac{2}{3}\sigma - \frac{1}{3}\xi , \qquad (6.08)$$

where we put for brevity

$$\sigma = \sqrt{\xi^2 + 3\zeta} , \qquad (6.09)$$

the root being taken positive.

The extremum value of the phase is equal to

$$\phi^* = \frac{1}{27}\left(4\sigma^3 - 3\sigma^2\xi - 2\xi^3\right) . \qquad (6.10)$$

In the following we shall always use the symbol ϕ^* to denote this extremum value. Applying the method of stationary phase we deduce for U^* the asymptotic expression

$$U^{*} = e^{i\phi^{*}} \sqrt{\frac{1}{3} - \frac{2}{3}\frac{\xi}{\sigma}} \, . \tag{6.11}$$

The integrand in V^{*} differs from that in U^{*} by a slowly varying factor, which is for large negative values of t approximately equal to

$$\frac{\left(w_2'/w_2\right) - q}{\left(w_1'/w_1\right) - q} = \frac{q - i\sqrt{-t}}{q + i\sqrt{-t}} \, . \tag{6.12}$$

Therefore the asymptotic value of V^{*} will differ from that of U^{*} by the factor (6.12) taken in the extremum point. Hence we have

$$V^{*} = e^{i\phi^{*}} \sqrt{\frac{1}{3} - \frac{2\xi}{3\sigma}} \; \frac{q - \frac{i}{3}(\sigma - \sigma\xi)}{q + \frac{i}{3}(\sigma - 2\xi)} \, . \tag{6.13}$$

Let us elucidate the geometrical meaning of the formulas obtained.

We consider the ray, which goes after reflection through the point x,y,z. Determining the coordinates x_o, y_o of the point of the surface, where the reflection took place, we obtain the following approximate formulas, valid for gliding incidence

$$x_o = x - s \; ; \quad y_o = y \, , \tag{6.14}$$

where

$$s = \frac{\sigma + \xi}{3am} \, . \tag{6.15}$$

Geometrically s is the length of the path, traversed by the ray after reflection. The cosine of the incidence angle is equal to

$$\cos\theta = -(ax_o + by_o) = \frac{1}{3m}(\sigma - 2\xi) . \tag{6.16}$$

The exact value of the difference $x_o - x + s$ is

$$x_o - x + s = 2s\cos^2\theta . \tag{6.17}$$

The phase difference of the reflected and the incident wave is proportional to this quantity. We have

$$\phi* - \phi = k(x_o - x + s) = 2\,ks\cos^2\theta . \tag{6.18}$$

Inserting in (6.18) the values of s and of $\cos\theta$ from (6.15) and (6.16) and using (4.11) we obtain

$$\phi* - \phi = \frac{2}{27}(\sigma + \xi)(\sigma - 2\xi)^2 . \tag{6.19}$$

It is easy to verify that (6.19) is equal to the difference of the quantities (6.10) and (6.01).

Hence the phase difference of the two terms in (6.02) and (6.03) is in agreement with the results obtained from geometrical optics.

Consider now the amplitude of the reflected wave.

Inserting (6.11) in (6.02) we shall have

$$U = e^{i\phi} - e^{i\phi*}\sqrt{\frac{1}{3} - \frac{2\xi}{3\sigma}} . \tag{6.20}$$

Using the expression (4.16) for q and the value (6.16) for $\cos\theta$ and inserting (6.13) in (6.03) we obtain

$$V = e^{i\phi} - e^{i\phi*}\sqrt{\frac{1}{3} - \frac{2\xi}{3\sigma}}\;\frac{1 - \cos\theta\sqrt{\eta}}{1 + \cos\theta\sqrt{\eta}} . \tag{6.21}$$

The function U corresponds to the case when the polarization of the incident wave is such that the electrical vector is perpendicular to the plane of incidence. The function V corresponds to the case of an electrical vector parallel to the plane of incidence. It is easy to see that in both cases our formulas give the correct values of the Fresnel coefficients.

7. CONCLUSION

The formulas obtained above give immediately the field in the vicinity of any point situated on the surface of a conducting body on the boundary of the geometrical shadow. Since this point may be chosen in an arbitrary way, our formulas give also the field in a certain ring-shaped region, adjacent to the closed line, which represents the boundary of the geometrical shadow on the surface (penumbra region). Consider now the field outside this region, but still near the surface (at distances from the surface, that are small as compared with its curvature radius). In the shaded part of this spatial region we may put the field amplitude equal to zero. Indeed the obtained solution decreases exponentially as the distance from the shadow boundary increases, and if the quantity $\xi + \sqrt{\zeta}$ is positive and large, this solution can be considered practically to be zero. We thus obtain a continuous transition to complete shadow. Let us now consider the illuminated region. In § 6 we have seen that in the remote part of the illuminated region our formulas give a field which coincides with that obtained from the Fresnel formulas. Hence it follows that if we use our formulas in the penumbra region and calculate the field with the help of Fresnel's formulas in the illuminated one, we shall obtain a continuous transition from penumbra to light.

In this way our formulas permit us to determine the field on and near the whole surface of the body (within a certain layer). Particularly, they give the current distribution, induced by an incident plane wave on the surface of the body. But if the current distribution is known, the field of the scattered wave can be determined in the whole space (also at large distances from the body) by applying well-known formulas for the vector-potential due to given currents.

As a final result our formulas give thus a complete (though approximate) solution of the problem of diffraction of a plane wave by a conducting convex body of arbitrary shape.

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VI. PROPAGATION OF THE DIRECT WAVE AROUND THE EARTH WITH DUE ACCOUNT FOR DIFFRACTION AND REFRACTION

V. A. Fock

Introduction

Having assumed the homogeneity of the earth's surface, the propagation of the radio waves around the earth is conditioned basically by the following three considerations: diffraction around the convex surface of the earth, refraction in the lower layers of the atmosphere, and reflection from the ionosphere. At short distances, of the order of a hundred or several hundred kilometers, the reflection from ionosphere plays no role. But at distances of the order of a thousand or several thousand kilometers the reflection from the ionosphere begins to play a substantial role, because the direct wave begins to have added to it the reflected waves which have substantially greater intensity than the direct wave.

However, even at these great distances it is possible, under certain conditions, to separate the direct wave and to observe it independently. Its study is of important practical interest for the interference methods of determining distances. For this reason the development of a theory which would give the amplitude and phase of the direct wave up to the ultimate distances, presents a very important problem for practical purposes.

p.126

The theory of direct wave must take account of both the diffraction and the refraction. Nevertheless, in view of the complexity of the task, in the majority of the theoretical investigations the atmospheric refraction either is not taken into the account at all or is treated very crudely, using methods of geometrical optics. The extremely important concept of the equivalent radius of the earth has not received adequate theoretical foundation in this case. The concept has been introduced on the basis of considerations of bent rays, and yet, in the region of the penumbra and particularly in the region of the umbra, the concept of ray as such loses its significance. In connection with this, those conditions under which the replacement of the earth's radius by the equivalent radius is permissible have not been made clear.

In this paper we shall give an approximate solution of the Maxwell's equations for the Hertzian vector which will take account of both the diffraction and the refraction. This solution is valid for very general assumptions regarding the variations of the index of refraction with height.

In certain practically important cases this solution may be expressed by functions introduced by us in our solution of the problem of propagation of radio waves in homogeneous atmosphere. These functions are partially tabulated; in those cases where there are tables the computation of the field with due account for refraction presents little work. Incidentally, we shall give the basis for the concept of the equivalent radius of the earth and shall show that this concept is applicable in the region of the umbra and penumbra (where the geometrical optics are not applicable) and shall make clear the conditions when the employment of the concept of an equivalent radius of the earth is permissible.

1. Differential Equations and The Boundary Conditions of The Problem

Let us designate by r, θ, and ϕ the spherical coordinates with the origin at the center of the earth's sphere and with the polar axis passing through the transmitting dipole. We shall assume the dipole to be located on the surface of the earth and we shall study the field in the air. The radius of the earth we shall designate by $\underline{a}$. The dielectric constant of the air we shall assume to be a function of height $h = r - a$ above the surface of the earth.

$$\epsilon = \epsilon(h), \quad h = r - a \tag{1.01}$$

As in the case of the homogeneous atmosphere, the component fields in the air may be expressed by Hertzian function U. We have:

$$E_r = \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial U}{\partial\theta}\right), \tag{1.02}$$

$$E_\theta = -\frac{1}{\epsilon r} \frac{\partial}{\partial r}\left(\epsilon r \frac{\partial U}{\partial\theta}\right), \tag{1.03}$$

$$H_\phi = -ik_o \epsilon \frac{\partial U}{\partial\theta} \tag{1.04}$$

whereas the remaining component fields are equal to zero. The time dependence of the field we express by $e^{-i\omega t}$ where

$$\frac{\omega}{c} = k_o = \frac{2\pi}{\lambda_o}. \tag{1.05}$$

Here λ_o is the wave length in free space (in our problem it is necessary to distinguish it from that in the air). The value of the dielectric constant of the air at the surface of the earth we shall denote by $\epsilon_o = \epsilon(0)$, and we shall denote by

$$k = \frac{2\pi}{\lambda} = k_o \sqrt{\epsilon_o} \tag{1.06}$$

the wave number evaluated at the surface of the earth.

The field, expressed by the formulas (1.02) and (1.04) will satisfy Maxwell's equations if the function U satisfies the equation

$$\frac{\partial}{\partial r}\left(\frac{1}{\epsilon}\frac{\partial}{\partial r}(\epsilon r U)\right) + \frac{1}{r \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial U}{\partial\theta}\right) + k_o^2 \, \epsilon r U = 0 . \tag{1.07}$$

Let us introduce a new function

$$U_1 = \epsilon r \sqrt{\sin\theta} \; U \tag{1.08}$$

This function must satisfy the equation:

$$\frac{\partial}{\partial r}\left(\frac{1}{\epsilon}\frac{\partial U_1}{\partial r}\right) + \frac{1}{\epsilon r^2}\left[\frac{\partial^2 U_1}{\partial\theta^2} + \left(\frac{1}{4} + \frac{1}{4\sin^2\theta}\right) U_1\right] + k_o^2 \, U_1 = 0 . \tag{1.09}$$

The field at the surface of the earth must satisfy Leontovich's conditions

$$E_\theta = -\frac{1}{\sqrt{\eta}} H_\phi , \tag{1.10}$$

where

$$\eta = \epsilon_2 + i\,(4\pi/\omega)'\,\sigma_2 \tag{1.11}$$

is the complex dielectric constant of the soil. Leontovich's condition will be satisfied if the function U_1 satisfies the condition

$$\frac{\partial U_1}{\partial r} = -\frac{i k_o \epsilon_o}{\sqrt{\eta}} U_1 \quad (\text{at } r = a) . \tag{1.12}$$

In the function U_1 let us separate out a rapidly varying factor by assuming

$$U_1 = e^{ika\theta} \; U_2' = e^{iks} \; U_2 \, , \tag{1.13}$$

where K has the significance (1.06) and $s = a\theta$ is the length of the arc along the earth's surface from the point where the dipole is located to the point where the field is being computed. For function U_2 we obtain the equation

$$\frac{\partial^2 U_2}{\partial r^2} + 2i \frac{k}{a} \frac{\partial U_2}{\partial \theta} + k^2 \left(\frac{\epsilon}{\epsilon_o} - \frac{a^2}{r^2} \right) U_2 =$$

$$= \frac{\epsilon'}{\epsilon} \frac{\partial U_2}{\partial r} - \frac{1}{r^2} \left[\frac{\partial^2 U_2}{\partial \theta^2} + \left(\frac{1}{4} + \frac{1}{r \sin^2 \theta} \right) U_2 \right] , \tag{1.14}$$

where ϵ' denotes the derivative of $\epsilon(h)=\epsilon(r-a)$ with respect to r.

The equation (1.14) is so written that the left-hand portion contains the most important terms while the right hand side contains corrective terms, which, as we shall show, may be replaced by zero.

Upon evaluating the order of magnitude of the resultant we may take advantage of the results obtained for the case of homogeneous atmosphere. If we introduce the large parameter

$$m = \sqrt[3]{\frac{ka}{2}} \tag{1.15}$$

Then

$$\frac{\partial U_2}{\partial r} = 0 \left(\frac{k}{m} U_2 \right) , \quad \frac{\partial U_2}{\partial \theta} = 0 \left(\frac{ka}{m^2} U_2 \right) , \tag{1.16}$$

where the symbol 0 stands for "of the order of".

On the other hand, if we exclude from our considerations the ionosphere (where ϵ may become zero) then the gradient of the logarithm of ϵ will be of the order of the curvature of the earth's surface so that

$$\frac{\epsilon'}{\epsilon} = O\left(\frac{1}{a}\right). \tag{1.17}$$

From this it is seen that separate terms of the left side of (1.14) will be of the order not less than $\frac{k^2}{m^2}\ U_2$, while on the right side the terms containing the derivatives, will be of the order $\frac{k^2}{m^4}\ U_2$. As regards the terms containing $\sin^2\theta$ in the denominator then under condition

$$ks \gg m \tag{1.18}$$

these terms likewise will be small. In this way, by dropping the magnitude of the order $\frac{1}{m^2}$ as compared with unity, we shall be able to substitute zero for the right side of equation (1.14) after which we shall obtain

$$\frac{\partial^2 U_2}{\partial r^2} + 2i\,\frac{k}{a}\,\frac{\partial U_2}{\partial \theta} + k^2\left(\frac{\epsilon}{\epsilon_o} - \frac{a^2}{r^2}\right)U_2 = 0\,. \tag{1.19}$$

This is a parabolic equation of our problem which resembles in form the Schroedinger equation of the quantum mechanics.

We can make further simplification in this equation by making use of the approximate equality

$$1 - \frac{a^2}{r^2} = 2\,\frac{h}{a}\,. \tag{1.20}$$

Introducing, in addition to that, in place of the angle θ the length of the arc $s = a\theta$ and regarding s and h as independent variables, we arrive at

$$\frac{\partial^2 U_2}{\partial h^2} + 2ik\,\frac{\partial U_2}{\partial s} + k^2\left(\frac{\epsilon - \epsilon_o}{\epsilon_o} + \frac{2h}{a}\right)U_2 = 0\,. \tag{1.21}$$

the boundary condition for U_2 at the surface of the earth will be the same as for U_1, namely

$$\frac{\partial U_2}{\partial h} = - ik \sqrt{\frac{\epsilon_c}{\eta}}\ U_2 \quad (\text{at } h = 0) \ . \tag{1.22}$$

The condition at infinity* ($h \to \infty$) maybe obtained from consideration of the phase of the Hertzian function. If we let

$$U = \left| U \right| e^{i\phi} \qquad U_2 = \left| U_2 \right| e^{i(\phi - ks)} \ , \tag{1.23}$$

Then, since we are considering the wave coming from the source, the phase of ϕ must increase with increase in height h. From this we obtain the condition

$$\frac{\partial \phi}{\partial h} > 0 \ , \tag{1.24}$$

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*Footnote:

We are taking an opportunity to correct an inaccuracy permitted in the discussion of the conditions at infinity in the article by M. A. Leontovich and V. A. Fock.[2] In this article during the solution of the problem for spherical earth there was set a requirement that not only the Hertzian function but also all separate items of the series representing it (independent partial solutions) remained finite with unlimited increase of the variable (proportional to the height h). Actually this requirement is not met. Nevertheless, the partial solutions were selected correctly and all of the remaining results of above article are also correct. The reasons given for the selection of the partial solutions must be replaced by the condition for the phase, analogous to our condition $\frac{\partial \phi}{\partial h} > 0$. Instead of that it was also permissible to require exponential attenuation of the wave in the presence of unlimited increase of the variable x, proportional to the horizontal distance s.

which must be fulfilled, at least for sufficiently large values of h.

In addition to the above requirements, the Hertzian function U, and also function U_2 must remain finite and continuous throughout the entire space with the exception of the region immediately adjacent to the source.

For a singular solution to the equation (1.21) it remains for us to formulate a condition which must be satisfied by function U_2 in the region immediate to the source. First, it is apparent that in the immediate neighborhood of the source equation (1.18) is invalid, and equation (1.21) itself is no longer correct. For this reason the region must, nevertheless, remain in the "wave zone". For example, we may take a region where a "reflection formula" applies, and obtain the desired condition by demanding that the sought solution in this region be in conformity with a reflection formula.

The reflection formula has the form

$$U = \frac{e^{ikR}}{R} (1 + f) , \tag{1.25}$$

where f is the Fresnel coefficient. Because we are making use of the boundary conditions of Leontovich (1.10) we thereby assume that $|\eta| >> 1$. If we, in addition, will assume that $h << s$, i.e., consider low angles of ray above the earth's surface, then we can assume

$$R = s + \frac{h^2}{2s} , \qquad f = \frac{h\sqrt{\eta} - s}{h\sqrt{\eta} + s} . \tag{1.26}$$

Substituting these expressions in (1.25), we come to the conclusion that in the region where the "reflection formula" is applicable the function

$$U_2 = e^{-iks} \quad \epsilon r \sqrt{\sin \theta} \; U \tag{1.27}$$

must be transformed into the form

$$U_2 = \frac{\epsilon_o\sqrt{a}}{\sqrt{s}} \; \frac{2h\sqrt{\eta}}{h\sqrt{\eta} + s} \; e^{i\frac{kh^2}{2s}} \tag{1.28}$$

Automatically, this condition is equivalent to the requirement that when $s \to 0$ and $h > 0$ the function U_2 has a property characterized by the condition

$$\lim_{s\to 0}\left(U_2 - \frac{2\epsilon_o\sqrt{a}}{\sqrt{s}} \; e^{i\frac{kh^2}{2s}}\right) = 0 \; . \tag{1.29}$$

More detailed basis for the condition (1.29) may be found in the referenced work by M. Leontovich and V. Fock.[2]

Let us note that in place of conditions (1.29) and (1.28) we could have set up still a more stringent condition, requiring that in that region where the influence of the curvature of the earth's surface and of the nonhomogeneity of the atmsophere already ceases and where the formula of Weyl-van der Pol is applicable*, our solution should pass into the solution by Weyl van der Pol.

2. Transfer to Dimensionless Quantities

The differential equation for function U_2, derived by us, takes the form:

$$\frac{\partial^2 U_2}{\partial h^2} + 2ik\frac{\partial U_2}{\partial s} + k^2\left(\frac{\epsilon - \epsilon_o}{\epsilon_o} + \frac{2h}{a}\right)U_2 = 0 \; . \tag{2.01}$$

Let us consider the coefficient of U_2 in this equation. Having denoted by ϵ_o' the value of the gradient of the dielectric

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* Footnote:

The range of application of the formula Weyl-van der Pol was investigated in detail in our work.[1]

constant near the surface of the earth, we can separate out of the expression for ϵ the linear term and express the coefficient of U_2 in the form

$$k^2\left(\frac{\epsilon-\epsilon_o}{\epsilon_o}+\frac{2h}{a}\right)=k^2\left[\frac{\epsilon-\epsilon_o-\epsilon_o' h}{\epsilon_o}+\left(\frac{2}{a}+\frac{\epsilon_o'}{\epsilon_o}\right)h\right]. \tag{2.02}$$

Now let

$$\frac{1}{a^*}=\frac{1}{a}+\frac{\epsilon_o'}{2\epsilon_o}. \tag{2.03}$$

The quantity (2.03) is the difference between the curvature of the earth's surface and the curvature of the ray, while the quantity a^* is commonly designated as the equivalent radius of the earth. Adopting the nomenclature of (2.03) we can write the formula (2.02) in the form

$$k^2\left(\frac{\epsilon-\epsilon_o}{\epsilon_o}+\frac{2h}{a}\right)=\frac{2k^2}{a^*}h\ (1+g), \tag{2.04}$$

where:

$$g=\frac{a^*}{2\epsilon_o}\left(\frac{\epsilon-\epsilon_o}{h}-\epsilon_o'\right). \tag{2.05}$$

As can be seen from (2.05) the quantity g is expressed in dimensionless units and depends upon the average gradient (averaged along the height) of the dielectric constant of the air** and the value of the gradient at the earth's surface. In the case of normal atmosphere the magnitude g is positive but in case of temperature inversion it may become negative and then only starting with a certain height will again become positive. The absolute magnitude of g is usually not greater than 0.2 or 0.3. With $h \to \infty$ the theoretical

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** Footnote:

Calculated from the surface of the earth to the given height.

significance of g becomes $\frac{a^* - a}{a}$ and with h = 0 will be g = 0. In the case of normal atmosphere the quantity g changes very slowly, but in case of inversion its change takes place considerably faster.

Substituting expression (2.04) in the differential equation (2.01) we obtain

$$\frac{\partial^2 U_2}{\partial h^2} + 2ik \frac{\partial U_2}{\partial s} + \frac{2k^2}{a^*} h \,(1 + g)\, U_2 = 0 \,. \qquad (2.06)$$

For investigation of equation (2.06) it is convenient to change from h and s to dimensionless quantities. For this purpose we shall introduce vertical and horizontal scales.

$$h_1 = \sqrt[3]{\frac{a^*}{2k^2}}\,, \qquad s_1 = \sqrt[3]{\frac{2a^{*2}}{k}} \qquad (2.07)$$

and we denote

$$\frac{h}{h_1} = y\,, \qquad \frac{s}{s_1} = x \qquad (2.08)$$

In order to simplify the condition (1.29), we will also change to a new dimensionless function W_1, assuming

$$U_2 = \frac{\epsilon_o \sqrt{a}}{\sqrt{s_1}}\, W_1 \,. \qquad (2.09)$$

In addition to that let

$$q = ikh_1 \sqrt{\frac{\epsilon_o}{\eta}} = i \sqrt[3]{\frac{ka^*}{2}} \sqrt{\frac{\epsilon_o}{\eta}} \,. \qquad (2.10)$$

Using the new notation, the differential equation, the boundary condition, and the condition determining singularity are written

$$\frac{\partial^2 W_1}{\partial y^2} + i\frac{\partial W_1}{\partial x} + y\,(1+g)\,W_1 = 0\,, \tag{2.11}$$

$$\frac{\partial W_1}{\partial y} + qW_1 = 0 \quad (\text{with } y = 0) \tag{2.12}$$

$$\lim_{x\to 0}\left(W_1 - \frac{2}{\sqrt{x}}\,e^{i\frac{y^2}{4x}}\right) = 0 \quad (y > 0) \tag{2.13}$$

In addition to that, there remains in force the condition for phase of $\phi = ks + \arg W_1$, namely

$$\frac{\partial \phi}{\partial y} > 0 \qquad (\text{with } y \gg 1) \tag{2.14}$$

The quantity g entering in the equation (2.11) was determined above [formula (2.05)] as a function of height h. Denote by h_o some height characterizing the rate of change of the gradient of the dielectric constant of the air, e.g., that height interval within which the gradient changes by e = 2.718 times. (For normal atmosphere h_o = 7400M; in other cases it is possible only to denote the order of magnitude of h_o which is all that we need.) The quantity g we may regard as a function of the ratio h/h_o.

$$g = g(h/h_o)\ , \quad g(0) = 0\ , \tag{2.15}$$

considering that the derivative of this function relative to its "argument" will be of the order of unity. With the transfer to the dimensionless quantities (2.08), we must regard g as a function of y. Since $h = h_1 y$, we shall have

$$g = g(\beta y) \tag{2.16}$$

where

$$\beta = \frac{h_1}{h_o} = \frac{1}{h_o} \sqrt[3]{\frac{a^*}{2k^2}} \tag{2.17}$$

In the future we shall regard the parameter β as a small quantity. In order to evaluate its order of magnitude assume h_o = 7400M (normal atmosphere) and replace the equivalent radius $\underline{a}^*$ by the geometrical radius $\underline{a}$. Then for λ = 1 M, λ = 100 M, and λ = 1000 M there shall be obtained correspondingly β = 0.006, β = 0.027, β = 0.13, β = 0.58. In case of inversion, the magnitude of h_o will be significantly less and the parameter β will become small only for proportionately shorter wavelengths.

3. Solution of Equations

If in the equation

$$\frac{\partial^2 W_1}{\partial y^2} + i \frac{\partial W_1}{\partial x} + y \left[1 + g(\beta y)\right] W_1 = 0 \tag{3.01}$$

we assume β = 0, because $g(0)$ = 0, the function g will likewise become equal to zero, and the equation will become the same as that which was discussed and solved [together with the boundary conditions (2.12) and (2.13)] in our previous work devoted to the investigation of the case of homogeneous atmosphere. However, it is important to note that the condition β = 0 corresponds not to the assumption of the homogeneity of the atmosphere, but to the more general assumption of constancy of gradient of the dielectric constant. The formulas obtained are the same as in the case of homogeneous atmosphere with the exceptions that in the expressions for x, y, and q in place of the radius of the earth $\underline{a}$, there is involved the equivalent radius $\underline{a}^*$. In this way, the smallness of the magnitude β determines the degree of exactness with which it is possible to employ (for finite values of y) the concept of the equivalent radius.

The solution for $\beta = 0$ was obtained by us in the form of an integral containing the complex Airy function. The latter is that solution of the differential equation

$$w''(t) = tw(t) \quad . \tag{3.02}$$

which has, for large negative values of t, the asymptotic expression

$$w(t) = e^{i\frac{\pi}{4}}(-t)^{-\frac{1}{4}} e^{i\frac{2}{3}(-t)^{3/2}} , \tag{3.03}$$

The solution for $g(\beta y) = 0$ has the form

$$W_1 = e^{-i\frac{\pi}{4}} \frac{1}{\sqrt{\pi}} \int_C e^{ixt} \frac{w'(t-y)}{w'(t) - qw(t)} dt , \tag{3.04}$$

where the contour ranges from $t = i\infty$ to $t = 0$ and from $t = 0$ to $t = \infty e^{i\alpha}$ $(0 < \alpha < \frac{\pi}{3})$ enclosing all of the roots of the denominator of the function under the integral. (This contour can, of course, be replaced by some other equivalent contour.) This solution coincides with that which was obtained earlier in our first paper.[1]

Using an analogous method we shall attempt to find a solution for our equations for the general case of $\beta \neq 0$. At the same time we shall not make the assumption that β is small and only later, with the aim of simplifying the obtained general solution, will we make use of a restriction regarding the smallness of β.

The equation (3.01) permits separation of the variables. Partial solutions of the equation (3.10) having the form of a function of x multiplying a function of y containing an arbitrary parameter t, will be written as

$$W_1 = e^{ixt} f(y, t) , \tag{3.05}$$

where f(y, t) satisfies the equation

$$\frac{d^2f}{dy^2} + \left[y - t + yg'(\beta y)\right] f = 0 . \tag{3.06}$$

From the theory of differential equations it is known that if the initial value (i.e. its value with y = 0) of the function f and its derivative with respect to y are entire functions of the parameter t, then the integral of the equation (3.06) will be an entire transcendental function of t. We shall designate by f(y, t) that integral of the equation (3.06) which is an entire transcendental function of t and permits, for large values of the difference y - t (or its real part), the asymptotic representation

$$f(y, t) = \frac{Ce^{i\frac{\pi}{4}}}{\sqrt[4]{y - t + yg(\beta y)}} \exp\left[i \int_{\tau}^{y} \sqrt{u - t + ug(\beta u)}\ du\right]. \tag{3.07}$$

The lower limit of τ in the integral which appears in the exponential may be taken arbitrarily. The coefficient C may be a function of parameter t. The phase factor $e^{i\frac{\pi}{4}}$ is added in order that, with g = 0 and τ = t, the expression (3.07) will transform into the asymptotic expression for the function

$$f(y, t) = Cw(t - y) . \tag{3.08}$$

The expression (3.07) was taken in accordance with the requirement $\frac{\partial\phi}{\partial y} > 0$, imposed on the phase.

Designating by f(y, t) the integral of equation (3.06) just determined, we shall consider the expression

$$W_1 = e^{-i\frac{3\pi}{4}} \frac{1}{\sqrt{\pi}} \int_{\Gamma} e^{ixt} \frac{f(y, t)}{\left(\frac{\partial f}{\partial y} + qf\right)_{y=0}} dt , \qquad (3.09)$$

Where the contour Γ has the form analogous to contour C in the integral (3.04).

First, let us note that the function under the integral is uniquely determined by the conditions laid down previously, because the factor C which remained unevaluated in (3.07) has been eliminated.

Further, the function under the integral in (3.09) represents a meromorphic function of the complex variable t; the only singular points in it are the roots of the denominator.

Investigation of the roots of the denominator in (3.09) is difficult to carry out with full rigor. For such investigation it is necessary to know the behavior of the function $g(\beta y)$ with complex values of y in the vicinity of arg $y = \frac{\pi}{3}$. However, on the basis of certain not fully rigorous considerations which we shall not cite here, it can be expected that if the function $g(\beta y)$ will remain small in the indicated complex region (e.g., $|g| < \frac{1}{2}$), then the roots will be located in the same way as in the case $g = 0$, i.e., in the first quadrant of the plane t in the vicinity of arg $t = \frac{\pi}{3}$. In any case it will be so for small values of parameter β.

It is also necessary for us to know the behavior of the function $f(y, t)$ for positive values of $t - y$ (and also in the certain sector of the t-plane including the positive real axis). The desired asymptotic expression will be obtained by the analytical continuation of expression (3.07) through the third and fourth quadrants of the plane t, because in the first are located the roots of $f(y, t)$. It will have the form

$$f(y, t) = \frac{C}{\sqrt[4]{t - y - yg\,(\beta y)}} \exp\left[\int_y^\tau \sqrt{t - u - ug\,(\beta u)}\;du\right]. \tag{3.10}$$

If we assume here that $g = 0$ and take $\tau = t$, then this expression will lead, as did (3.07), to the asymptotic expression for the function (3.08).

Knowing the location of the roots and the behavior of the function under the integral on both sides of the region where the roots are located, it is then possible to take in the integral (3.09) the contour Γ in such a way that it includes all the roots of the denominator and goes away with branches to infinity. For the initial branch of the contour (disappearing into infinity) will hold correctly the asymptotic relation (3.07) and for the terminal branch (disappearing into infinity)- the expression (3.10). At the same time the integral taken along this contour will be converging.

The preceding discussion had the purpose to show that the expression (3.09) for the function W_1 has a definite mathematical significance.

Let us show now that it satisfies all conditions which have been laid down. First, it is clear that it satisfies the differential equation (3.01) because it is satisfied by the function under the integral. Further, it satisfies the boundary condition (2.12).

$$\frac{\partial W_1}{\partial y} + qW_1 = 0 \qquad \text{with } y = 0 \tag{3.11}$$

In fact, by differentiating in (3.09) under the sign of the integral and then assuming $y = 0$, we shall see that the numerator of the fraction will cancel with the denominator and the function under the integral will be holomorphic, for which reason the integral will become equal to zero. Then the integral will become converging and, therefore, finite for all positive values of x and y. It is not difficult to verify that it will satisfy the condition for the phase $\left(\frac{\partial\phi}{\partial y} > 0\right)$.

It remains for us to check whether the expression (3.09) has the singularity near x = 0, which is required by the condition (2.13), or, what is equivalent, to verify whether at short distances from the source it gives the Weyl van der Pol formula or the reflection formula.

With the aid of the asymptotic expression (3.07) and (3.10) for f(y, t), it is possible to show that if x and y are small, and the relation $\frac{y}{x}$ is large, then the principal portion of the integration will lie in the region of large negative values of t. (The earlier contour can be deformed so that it passes through this region.) Making use of the expression (3.07), we obtain for large negative values of t:

$$\frac{f(y,\ t)}{f(0,\ t)} = \sqrt[4]{\frac{-t}{y - t + yg\ (\beta y)}}\ \exp\left[i \int_0^y \sqrt{u - t + ug}\ du\right]. \tag{3.12}$$

From this

$$\frac{1}{f}\frac{\partial f}{\partial y} = i\ \sqrt{y - t + yg\ (\beta y)} \tag{3.13}$$

$$\left(\frac{1}{f}\frac{\partial f}{\partial y} + q\right)_{y\ =\ 0} = i\ \sqrt{-\ t} + q\ . \tag{3.14}$$

But when y is small the term $yg(\beta y)$ is small compared with y and we can write in place of (3.12)

$$\frac{f(y,\ t)}{f(0,\ t)} = \sqrt[4]{\frac{-\ t}{y\ -\ t}}\ \exp\left[i \int_0^y \sqrt{u\ -\ t}\ du\right]. \tag{3.15}$$

Let us note now that the same asymptotic expressions will be obtained for the same region if in place of $f(y, t)$ we substitute

$$f(y, t) = w(t - y) . \tag{3.16}$$

But after such substitution the integral (3.09) will transform into (3.04) and the latter gives, for small values of x, y the Weyl-van der Pol formula, the reflection formula, and the boundary condition (2.13).

We can also verify this more directly. Introducing the variable of integration $p = \sqrt{-t}$ and neglecting the quantities y and y^2 as compared with p we find that

$$\frac{f(y, -p^2)}{f(0, -p^2)} = e^{iyp} \tag{3.17}$$

and

$$\left(\frac{1}{f}\frac{\partial f}{\partial y} + q\right)_{y=0} = ip + q \tag{3.18}$$

Substitution of these quantities in the integral (3.09) gives

$$W_1 = e^{-i\frac{3\pi}{4}} \frac{2}{\sqrt{\pi}} \int_{\Gamma_2} e^{-i(xp^2 - yp)} \frac{p\,dp}{p - iq} , \tag{3.19}$$

where the contour Γ_2 intersects the positive real axis in the plane of p from below upwards (in the vicinity of point $p = \frac{y}{2x}$) If we should compute the integral (3.19) without neglecting anything, we shall arrive at the Weyl-van der Pol formula. If we compute it by the method of stationary phase we arrive at the reflection formula. If we neglect the quantity $|q|$ in comparison with $\frac{y}{2x}$, we obtain an expression which will reduce to zero the left side of (2.13) even before taking the limit.

By this it is proved that the expression (3.19) for W_1 represents the desired solution of our problem.

4. Investigation of The Solution for The Region of Direct Visibility

Instead of function W_1, it is more convenient to consider another function distinguished from W_1 by a factor $\sqrt{x}$. We shall let

$$V(x, y, q) = e^{i\frac{3\pi}{4}} \sqrt{\frac{x}{\pi}} \int_{\Gamma} e^{ixt} \frac{f(y, t)}{\left(\frac{\partial f}{\partial y} + qf\right)_0} dt . \qquad (4.01)$$

Remembering the connections between the functions U, U_1, U_2, and W_1, given by the formulas (1.08), (1.13), and (2.09), and neglecting the distinction between r and a and between ϵ and ϵ_0 when these quantities enter in the role of factors for U we can write

$$U = \frac{e^{iks}}{\sqrt{as \sin \frac{s}{a}}} V(x, y, q) , \qquad (4.02)$$

where s, as before is the horizontal distance, measured along the arc of the earth's surface, and x, y, and q, are connected with s, h, η by the relations

$$x = \frac{s}{s_1} , \quad y = \frac{h}{h_1} , \quad q = i \sqrt[3]{\frac{ka^*}{2}} \sqrt{\frac{\epsilon_0}{\eta}} , \qquad (4.03)$$

where

$$s_1 = \sqrt[3]{\frac{2a^{*2}}{k}} , \quad h_1 = \sqrt[3]{\frac{a^*}{2k^2}} . \qquad (4.04)$$

If s is small compared with the radius of the earth, then instead of $\sin \frac{s}{a}$ it is permissible to write simply $\frac{s}{a}$ (as it is usually written). However, since the formulas remain correct up to very great distances where the difference between the sine and the arc become significant, we retain $\sin \frac{s}{a}$ under the radical in (4.02).

The function $V(x, y, q)$ may be called the attenuation factor; in those cases where it is permissible to consider $\varepsilon = 0$ and to make use of the concept of the equivalent radius, equation (4.01) for V transforms into

$$V(x, y, q) = e^{-i\frac{\pi}{4}} \sqrt{\frac{x}{\pi}} \int_{\Gamma} e^{ixt} \frac{w(t - y)}{w'(t) - qw(t)} dt . \tag{4.05}$$

The function (4.05) was investigated in detail in our paper[1] and partially tabulated (for $q = 0$).

The investigation which follows will in many respects parallel the similar investigation in our paper.[1]

In the present section we shall regard the line-of-sight region which corresponds to section VI of our paper.[1]

Geometrical optics is valid in the line-of-sight region remote from the horizon. If we make use of the expression (3.12) and introduce the variable of integration $p = \sqrt{-t}$, we shall obtain for V an integral of the form

$$V = e^{-i\frac{3\pi}{4}} \frac{2}{\sqrt{\pi}} \sqrt{x} \int e^{i\omega} \left(\frac{p^2}{y + p^2 + yg\ (\beta y)}\right)^{\frac{1}{4}} \frac{pdp}{p - iq}, \tag{4.06}$$

where for brevity we denote

$$\omega = -xp^2 + \int_0^y \sqrt{u + p^2 + ug\ (\beta y)}\ du . \tag{4.07}$$

(Translator's Note: Do not confuse this use of ω for phase with the use of ω for angular frequency in the time dependence $e^{-i\omega t}$).

Computing the integral by the method of stationary phase, we find the extremal of the phase

[Translator's Note: Condition that $\frac{\partial\omega}{\partial p} = 0$ in (4.07)].

$$x = \frac{1}{2} \int_0^y \frac{du}{\sqrt{u + p^2 + ug\ (\beta u)}} \tag{4.08}$$

and after several operations we arrive at the expression

$$V = e^{i\omega} \frac{2p}{p - iq} \sqrt{2x \frac{\partial p}{\partial y}} \tag{4.09}$$

In this formula p represents a function of x and y determined from equation (4.08). For g = 0 and also for small values of x and y.

$$p = \frac{y - x^2}{2x}, \tag{4.10}$$

and the expression under the sign of the radical in (4.09) becomes equal to unity.

Formula (4.09) is valid also in the case where the magnitude of p is large and positive .

Our formulas permit a simple discussion from the point of view of geometrical optics. Actually the complete phase

$$\phi = ks + \omega \tag{4.11}$$

of the function J represents a solution of the eikonal equation

$$\left(1 + \frac{h}{a}\right)^2 \left(\frac{\partial\phi}{\partial h}\right)^2 + \left(\frac{\partial\phi}{\partial s}\right)^2 = k^2 \left(1 + \frac{h}{a}\right)^2 \frac{\epsilon}{\epsilon_0}, \tag{4.12}$$

which, after neglecting small quantities leads to the following equation for ω:

$$\left(\frac{\partial\omega}{\partial h}\right)^2 + 2k \frac{\partial\omega}{\partial s} = \frac{2k^2}{a^*} h(1 + g), \tag{4.13}$$

where to the right is the quantity (2.04). After transfer to the variables x and y we obtain from (4.13)

$$\left(\frac{\partial\omega}{\partial y}\right)^2 + \frac{\partial\omega}{\partial x} = y + yg'(\beta y) \ . \qquad (4.14)$$

Relationship (4.08) is an equation of the trajectory of the ray passing through the origin of the coordinates, and the quantity p is the parameter of this trajectory. The geometrical significance of the parameter p is:

$$p = \sqrt[3]{\frac{ka^*}{2}} \cos\alpha \ , \qquad (4.15)$$

where α is the angle between the ray and the vertical line in the vicinity of the source. The complete phase ϕ is the optical length of the path of the ray, reckoned from the source to the point x, y. The quantity $\frac{2p}{p - iq}$ is equal to

$$\frac{2p}{p - iq} = 1 + f \ , \qquad (4.16)$$

where f is Fresnel coefficient.

Thus, in those cases where geometrical optics is applicable, our formulas transform into the formulas of the geometrical optics.

Formula (4.09) is applicable for the ultimate values of x and y in that case where parameter p is positive and large. If x and y are small the following condition becomes necessary

$$\frac{y^2}{4x} = \frac{kh^2}{2s} \gg 1 \ . \qquad (4.17)$$

If the condition (4.17) is not fulfilled, in the case of small values of x and y and large values of p, the expression (4.06) remains in force, but the integral must be calculated differently, namely ω must be replaced by $- xp^2 + yp$ and the fourth root must be replaced by unity, after which the integral is reduced to form (3.19) (with a factor $\sqrt{x}$) and will give the Weyl-van der Pol formula.

Let us note that if x and $\sqrt{y}$ are large, and the parameter p small compared with these quantities, then the equation of the trajectory (4.08) may be solved approximately for p. We shall have an approximation

$$p = \frac{1}{2} \int_0^y \frac{du}{\sqrt{u + ug\,(\beta u)}} - x \qquad (4.18)$$

Under the same conditions

$$\omega = \omega_0(y) + \frac{1}{3} p^3 , \qquad (4.19)$$

where

$$\omega_0(y) = \int_0^y \sqrt{u + ug\,(\beta u)}\; du , \qquad (4.20)$$

and the symbol p must be interpreted as an abbreviated designation for quantity (4.18).

The equation p = 0 gives the geometrical boundary of the shadow. If the right part (4.18) becomes negative, then the equation (4.08) will not have a real answer for p; however, function (4.18) [and also (4.10)] retains significance also in this case. This apparent discrepancy is explained by the fact that the right-hand part of (4.08) is not an analytical function of p near the region p = 0.

The expressions (4.18) and (4.19) will be encountered by us in the region of the penumbra where geometrical optics is no longer applicable.

Investigation of The Solution for The Region of The Penumbra (Finite X and Y)

The region of the penumbra is characterized by the fact that within it the parameter p, determined by the formula (4.10) is either a positive or a negative quantity of the order of unity.

If x and y are finite we may construct a series for V, arranged according to poles of the function within the integral sign.

We shall have

$$V(x, y, q) = e^{i\frac{\pi}{4}} 2\sqrt{\pi x} \sum_{n=1}^{\infty} \frac{e^{ixt_n}}{D(t_n)} \frac{f(y, t_n)}{f(0, t_n)} , \qquad (5.01)$$

where

$$D(t) = - \frac{1}{f(0, t)} \left(\frac{\partial^2 f}{\partial y \partial t} + q \frac{\partial f}{\partial t} \right)_{y = 0} , \qquad (5.02)$$

and t represents a root of the equation

$$\left(\frac{\partial f}{\partial y} + qf \right)_{y = 0} = 0 . \qquad (5.03)$$

If β is not small then the computations using these formulas is extremely complicated. For this reason in the future we shall limit ourselves to the case of very small values of β. At the same time, however, we shall not consider as being small the product βy, but shall also consider large values of y (of the order $1/\beta$ and larger).

If β is small, then in computing the first roots of the function (5.03) we can replace $g(\beta y)$ by a linear function

$$g(\beta y) = \left[\beta g'(0)\right] y = \beta_0 y . \qquad (5.04)$$

The physical significance of the coefficient β_o is

$$\beta_o = h_1 \left(\frac{dg}{dh}\right)_o = \frac{h_1 a^{*} \epsilon_o''}{4\epsilon_o} , \tag{5.05}$$

where h_1 is the scale of height and ϵ_o'' is the value of the second derivative of ϵ with respect to height at the surface of the earth.

For small values of β_o and finite values of y and t in the role of the solution of the equation (3.06) we can take the function

$$f(y, t) = w(t - y) - \frac{\beta_o}{15} \Big[(3y + 2t)\, w(t - y) + $$
$$ + (3y^2 + 4yt + 8t^2)\, w'(t - y) \Big] . \tag{5.06}$$

Substituting this expression in (5.03) we find for the desired root the approximate expression

$$t_n = t_n^o + \frac{\beta_o}{15} \left[8(t_n^o)^2 - \frac{3 + 4t_n^o q}{t_n^o - q^2} \right] , \tag{5.07}$$

where t_n^o is the root of the equation

$$\omega'(t_n^o) - q\omega(t_n^o) = 0 . \tag{5.08}$$

which was investigated in detail in reference [1]. For function D(t) there is obtained the expression

$$D(t) = (t - q^2)\left(1 - \frac{4}{3}\beta_o t\right) + \frac{2}{3}\beta_o q . \tag{5.09}$$

The height coefficients encountered in the formula (5.01)

$$f_n(y) = \frac{f(y, t_n)}{f(0, t_n)} \tag{5.10}$$

may be obtained by numerical integration of the differential equation

$$\frac{d^2f_n}{dy^2} + \left[y - t_n + yg\ (\beta y)\right] f_n = 0 \qquad (5.11)$$

for the initial conditions

$$f_n(0) = 1 \quad \text{and} \quad f_n'(0) = -q\ . \qquad (5.12)$$

As long as y is finite (even though x may be very large) the values of V(x, y, g) obtained in this way will, for small values of β, differ but little from values for $\beta = 0$. More or less significant difference may appear only in the coefficient e^{ixt_n}, giving the attenuation and added phase. For this reason it is sufficient to apply the correction to these coefficients.

If no special accuracy is required, it is possible to neglect this correction and simply accept that in the case under consideration the expression for V(x, y, q) coincides with the one derived for the case of homogeneous atmosphere (under the condition that the radius of the earth is replaced by the equivalent radius). It is then possible to make use of all the formulas and tables obtained for that case.

VI. Investigation of The Solution for The Region of Penumbra (Large Values of X and Y)

The case presenting the greatest, from the practical standpoint, interest is the one where x and y are very large while the quantity

$$p = \frac{1}{2} \int_0^y \frac{du}{\sqrt{u + ug\ (\beta u)}} - x \qquad (6.01)$$

is finite. We already pointed out that the significance of p = 0 corresponds to the limit of direct visibility, where positive values of p correspond to the region of line of sight an the negative values of p to the region beyond the horizon.

In this case, in computing the integral (4.01) for V(x, y, q) it is necessary to keep in mind that the principal sector of integration will correspond to the finite values of t where y is large. For this reason it is necessary to find such analytical expression for f(y, t) which would be valid both for very large and for finite values of y - t. This is found to be possible for the condition of small value of β.

Let us introduce the quantity X, defined by the equation

$$\frac{2}{3}(-X)^{3/2} = \int_{\tau}^{y} \sqrt{u - t + ug(\beta u)}\, du \tag{6.02}$$

or

$$\frac{2X^{3/2}}{3} = \int_{y}^{\tau} \sqrt{t - u - ug(\beta u)}\, du\ , \tag{6.03}$$

where τ is the root of the equation

$$\tau - t + \tau g(\beta\tau) = 0\ . \tag{6.04}$$

For small values of β_o and for finite values of y and t.

$$X = t - y - \frac{\beta_o}{15}(3y^2 + 4ty + 8t^2)\ . \tag{5.05}$$

Then the function

$$f(y, t) = \sqrt{-\frac{dy}{dX}}\ \omega(X) \tag{6.06}$$

will present the solution of the equation (3.06) with an error of the order of β^2 for finite values of x and y of the order of

$\beta^{3/2}$, for large values of y and finite values t. With the aid of the expression (6.05) it is not difficult to prove that in expanding (6.06) according to the powers of β_0, the terms of the series up to β_0 inclusive are identical with (5.06). However, the expression (6.06) is valid in those cases when (for large values of y) the expansion of (5.06) is not applicable. If the quantity X is very large and negative (which takes place for large values of y) then the expansion (6.06) is transformed into the following:

$$f(y, t) = \frac{e^{i\frac{\pi}{4}}}{\sqrt[4]{y - t + yg(\beta y)}} \exp\left[i \int_{\tau}^{y} \sqrt{u - t + ug(\beta u)}\, du \right]. \tag{6.07}$$

The latter coincides with (3.07) if, in that equation, one makes C = 1 and takes for τ the root of the equation (6.04). In this way, through the use of the formula (6.06) we have verified that the same solution of the equation (3.06) will have, for finite values of y, the expression (5.06) and for large values of y, the expression (6.07).

We can now in evaluating the integral

$$V(x, y, q) = e^{i\frac{3\pi}{4}} \sqrt{\frac{x}{\pi}} \int_{\Gamma} e^{ixt} \frac{f(y, t)}{\left(\frac{\partial f}{\partial y} + qf\right)_0}\, dt \tag{6.08}$$

make use of both expressions (5.06) and (6.07) at the same time, namely, substitute the expression (6.07) in the numerator and expression (5.06) in the denominator. At the same time we can to some extent simplify both expressions. Neglecting minor corrections, we shall, in place of (5.06), write simply

$$f(y, t) = w(t - y) . \tag{6.09}$$

and in the formula (6.07) in the coefficient before the exponential function we shall neglect the quantity t as compared with y, and replace the exponent by the approximate expression

$$\int_{\tau}^{y} \sqrt{u - t + ug(\beta u)}\, du = \int_{0}^{y} \sqrt{u + ug(\beta u)}\, du - \frac{1}{2} t \int_{0}^{y} \frac{du}{\sqrt{u + ug(\beta u)}} . \qquad (6.10)$$

Using the notation of (4.18) and (4.20) we can write

$$f(y, t) = \sqrt{2 \frac{\partial p}{\partial y}}\, e^{i \frac{\pi}{4}}\, e^{i\omega_o y - it(x + p)} . \qquad (6.11)$$

As a result we are replacing the function f(y, t) in the denominator by the Airy function, and in the numerator by the exponential function.

Substituting (6.09) and (6.11) in the integral (6.08) we will obtain

$$V(x, y, q) = e^{i\omega_o(y)} \sqrt{2x \frac{\partial p}{\partial y}}\, \frac{1}{\sqrt{\pi}} \int_{\Gamma} e^{-ipt} \frac{dt}{w'(t) - qw(t)} . \qquad (6.12)$$

The remaining integral can be evaluated by a known function. In our work [1] it is denoted by

$$V_1(-p, q) = \frac{1}{\sqrt{\pi}} \int_{\Gamma} e^{-ipt} \frac{dt}{w'(t) - qw(t)} \qquad (6.13)$$

and investigated in detail. For the cases $q = 0$ and $q = \sqrt{i}$ there are tables.

Footnote:
The tables for $q = 0$ are published in [3].

Formula (6.12) gives the coefficient of attenuation for the region close to the horizon. It is interesting to compare this formula with the formula (4.09) valid in the region where geometrical optics are applicable. Making use of (4.19) we shall write the expression (4.09) in the form

$$V = e^{i\omega_0(y)} \sqrt{2x \frac{\partial p}{\partial y}} \frac{2p}{p - iq} e^{\frac{i}{3} p^3} . \tag{6.14}$$

But in our work [1] it was shown that the function (6.13) has, for large positive values of p, the asymptotic expression

$$V_1(- p, q) = \frac{2p}{p - iq} e^{\frac{i}{3} p^3} . \tag{6.15}$$

In this way our formula (6.12) is transformed in the line-of-sight region into the formula of the geometric optics.

For negative values of p the expression for $V_1(- p, q)$ may be written in the form

$$V_1(- p, q) = i2\sqrt{\pi} \sum_{n = 1}^{n = \infty} \frac{e^{-ipt_n}}{(t_n - q^2) w(t_n)} . \tag{6.16}$$

Where $|p|$ is large ($p < 0$) this series is reduced to the first term which gives the attenuation of the wave in the region of umbra according to the exponential law.

Function $V_1(- p, q)$ was first introduced in our works devoted to the diffraction by a body of arbitrary form. In these works there was established a principle of the local field in the region of the penumbra and it was shown that in that region the field is expressed by the function $V_1(- p, q)$ having a universal character.

The comparison of the formulas (6.12) and (6.14) allows us to say in a certain sense, that the wave reaches the horizon with amplitude and phase corresponding to the laws of geometrical optics for unlimited mediums and at the horizon suffers diffraction according to the law of local field in the region of the penumbra.

This picture is found to be in complete agreement with the ideas of L. I. Mandelstam in that in the propagation of electromagnetic waves along the surface of the earth the properties of the ground are significant not along the entire trajectory of the ray, but only in that region where there is located on the ground the transmitter and the receiver ("line of departure" and "line of arrival" area).

If we accept this picture then the solution obtained in this section may be applied to that case where the properties of the earth's surface in different areas are not equal, under the condition that in the function $V_1(-p, q)$ the complex parameter q corresponds to the properties of the ground in that area where the ray touches the earth.

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VII.

THEORY OF RADIOWAVE PROPAGATION IN AN INHOMOGENEOUS ATMSOPHERE FOR A RAISED SOURCE

V. A. Fock

Introduction

We have developed the theory of radiowave propagation in an atmosphere with dielectric constant dependent on height [1] for the case when the source is a vertical electric dipole situated on the earth's surface. On the other hand, we have considered [2] the case of a raised source (horizontal and vertical electric and magnetic dipoles) assuming a homogeneous atmosphere.

The formulas derived in [1] for the general case of arbitrary behavior of the refraction index, were developed there in more detail assuming normal refraction when the radiowave propagation has the same qualitative character as in a homogeneous atmosphere. The case of super refraction, when the lower layer of the atmosphere acquires the character of a wave guide, is of independent interest and merits special consideration. In the present work, we consider this case in detail. For its qualitative characteristics, the analogy with the unsteady problem in quantum mechanics of the dispersion of a wave packet in a given force field appears to be useful: apparently, this analogy has not been observed until now.

The question of radiowave propagation under the conditions when the atmsophere acts as a wave guide was also studied by

P. E. Krasnushkin by applying the normal mode method to planar-layered and spherical-layered media [3]. However, the interesting study of P. E. Krasnushkin has a predominantly qualitative character and a number of essential mathematical problems remain unexplained; in particular, the question of the spectrum of the complex eigenvalues of the "normal waves" and the boundary conditions for the corresponding "normal functions".

In Sec. 1 of the present work the fundamental equations and the boundary conditions of the problem are set down. In Sec. 2 the approximate form of the equations is considered (Leontovich's parabolic equations) with the corresponding boundary conditions and the conditions determining the singularity. In Sec. 3 an analogy is carried out between the formulated problem and the unsteady problem of quantum mechanics. After transformation to nondimensional quantities (Sec. 4), a study is made of the properties of the particular solutions of the differential equations (Sec. 5), from which there is then constructed a general solution in the form of a contour integral and a series (Sec. 6). The general theory is applied then to the case of super refraction (Sec. 7) where an example is considered in which the curve of the reduced refraction index is assumed to be composed of two rectilinear segments. In the last section (Sec. 8) there are derived approximate formulas, analogous to the semi-classical quantum mechanics formulas, for the determination of the attenuation coefficients and the height factors. Questions on numerical computation methods are not touched upon in this work.

Section 1. Fundamental Equations and Limiting Conditions

Let us denote by r, θ, ϕ, the spherical coordinates with origin at the center of the earth and with the polar axis passing through the radiating dipole. Let us denote the earth's radius by a. We assume the dipole to be found at a height $h' = b - a$ above the earth's surface so that its coordinates will be $r = b$, $\theta = 0$. We will consider the dielectric constant of the air, ϵ, to be a function of the height $h = r - a$, above the earth's surface.

The field in air can be expressed according to the well-known formulas through the Debye potentials u, v.

We have

$$\left.\begin{aligned} E_r &= \frac{1}{r}\Delta^* u \\ E_\theta &= -\frac{1}{\epsilon r}\frac{\partial^2(\epsilon r u)}{\partial r\,\partial\theta} + \frac{i\omega}{c\sin\theta}\frac{\partial v}{\partial\phi} \\ E_\phi &= -\frac{1}{\epsilon r\sin\theta}\frac{\partial^2(\epsilon r u)}{\partial r\,\partial\theta} - \frac{i\omega}{c}\frac{\partial v}{\partial\theta} \end{aligned}\right\} \qquad (1.01)$$

$$\left.\begin{aligned} H_r &= -\frac{1}{r}\Delta^* v \\ H_\theta &= \frac{i\omega}{c}\frac{\epsilon}{\sin\theta}\frac{\partial u}{\partial\phi} + \frac{1}{r}\frac{\partial}{\partial r}\frac{(rv)}{\partial\theta} \\ H_\phi &= -\frac{i\omega}{c}\frac{\partial u}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial^2(rv)}{\partial r\,\partial\phi} \end{aligned}\right\} \qquad (1.02)$$

The same expressions are applicable for the field within the earth if we understand by ϵ the complex dielectric constant of the earth. The dependence on time is assumed here in the form of the factor $e^{-i\omega t}$. The symbol Δ^* denotes the Laplace operator on a sphere:

$$\Delta^* u = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} \qquad (1.03)$$

Maxwell's equations will be satisfied if the functions u and v satisfy:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{\epsilon}\frac{\partial(\epsilon r u)}{\partial r}\right)+\frac{\Delta^{*} u}{r^{2}}+\frac{\omega^{2}}{c^{2}}\,\epsilon u = 0 \qquad (1.04)$$

and

$$\frac{1}{r}\frac{\partial^{2}(rv)}{\partial r^{2}}+\frac{\Delta^{*} r}{r^{2}}+\frac{\omega^{2}}{c^{2}}\,\epsilon v = 0 \qquad (1.05)$$

The continuity of the tangential components of the field will be guaranteed if the quantities

$$\epsilon r u\ ,\ \frac{1}{\epsilon}\frac{\partial(\epsilon r u)}{\partial r}\ ,\ rv\ ,\ \frac{\partial(rv)}{\partial r} \qquad (1.06)$$

are continuous.

By means of well-known reasoning, there is obtained the approximate form of the boundary conditions (Leontovich conditions). If we put $k = \frac{\omega}{c}$, denote the complex dielectric constant of the earth by η and keep ϵ for the dielectric constant of air, then we will have

$$\frac{\partial(\epsilon r u)}{\partial r} = -\,ik\,\frac{\epsilon}{\sqrt{\eta}}\,(\epsilon r u) \qquad (\text{for } r = a) \qquad (1.07)$$

and

$$\frac{\partial(rv)}{\partial r} = -\,ik\,\sqrt{\eta}\,(rv) \qquad (\text{for } r = a) \qquad (1.08)$$

Later we shall call a field for which $u \neq 0$, $v = 0$ "vertically polarized" and the field for which $u = 0$, $v \neq 0$ "horizontally polarized" In this sense, the field of a vertical electric dipole remains vertically polarized in all space. The field of a vertical magnetic dipole (horizontal frame) has horizontal polarization everywhere. A horizontal electric dipole excites fields of both forms: both horizontally and vertically polarized. In the case of a homogeneous atmosphere, the vertically polarized field decreases with increasing distance more slowly than a horizontally polarized. Consequently, the field from a horizontal electric dipole at small distances from the source will be predominantly horizontally polarized, but at large distances (in the region far beyond the horizon) the polarization will be predominantly vertical.

The vertically polarized field can be expressed through the function U (the Hertz function of a vertical electric dipole) which has the following properties: U satisfies the same differential equation (1.04) and the same boundary conditions (1.07) as u and has, near the source, a singularity of the form

$$U = \frac{e^{ikR}}{R} + U^{*} \tag{1.09}$$

where U^{*} remains finite, and

$$R = \sqrt{r^2 + b^2 - 2rb\cos\theta}\ ;\ k = \frac{\omega}{c} \tag{1.10}$$

Similarly, the horizontally polarized field can be expressed through the function W(the Hertz function of a vertical magnetic dipole) which satisfies the same differential equation (1.05) and the same boundary conditions (1.08) as v and has near the source a singularity of the form

$$W = \frac{e^{ikR}}{R} + W^{*} \tag{1.11}$$

where W^{*} remains finite.

The fields of the vertical and horizontal electric and magnetic dipoles with moment M are expressed through the functions U and W defined above.

For the vertical electric dipole we put

$$u = \frac{M}{b} U \; ; \quad v = 0 \tag{1.12}$$

For the vertical magnetic dipole (horizontal loop) we have

$$u = 0 \; ; \quad v = \frac{M}{b} W \tag{1.13}$$

For the horizontal electric dipole directed along the x axis which enters into (1.01) and (1.02), the functions u and v are determined from:

$$\begin{aligned} \Delta^{*} u &= - M \frac{\partial}{\partial \theta} \left(\frac{\partial U}{\partial b} + \frac{U}{b} \right) \cos \phi \\ \Delta^{*} v &= - ikM \frac{\partial W}{\partial \theta} \sin \phi \end{aligned} \tag{1.14}$$

where Δ^{*} is the Laplace operator on a sphere (1 03).

Finally, for the horizontal magnetic dipole directed along the x axis we have:

$$\begin{aligned} \Delta^{*} u &= - ikM \frac{\partial U}{\partial \theta} \sin \phi \\ \Delta^{*} v &= M \frac{\partial}{\partial \theta} \left(\frac{\partial W}{\partial b} + \frac{W}{b} \right) \cos \phi \end{aligned} \tag{1.15}$$

Therefore, in all four cases the study of the field reduces to the study of the functions U and W.

Section 2. Approximate Form of the Equations

Turning to the approximate form of the equations, let us denote by ϵ_1 the value of the dielectric constant of air near the source (in practice we can put $\epsilon_1 = 1$) and let us put

$$s = a\theta \tag{2.01}$$

such that s is the horizontal distance between the source and the point of observation, measured along the arc.

Instead of U and W let us introduce the slowly varying functions U_2 and W_2 by putting

$$U = \frac{\epsilon_1 e^{iks}}{\epsilon r \sqrt{\sin\theta}} U_2 \tag{2.02}$$

and

$$W = \frac{e^{iks}}{r\sqrt{\sin\theta}} W_2 \tag{2.03}$$

As shown in [1], after neglecting small quantities the equation in U_2 becomes

$$\frac{\partial^2 U_2}{\partial h^2} + 2ik\frac{\partial U_2}{\partial s} + k^2\left(\epsilon - 1 + \frac{2h}{a}\right)U_2 = 0 \tag{2.04}$$

Instead of r and θ, the quantities h (height) and s (horizontal distance) are taken as independent variables. In our approximation, the equation for W_2 will have the same form; viz.,

$$\frac{\partial^2 W_2}{\partial h^2} + 2ik\frac{\partial W_2}{\partial s} + k^2\left(\epsilon - 1 + \frac{2h}{a}\right)W_2 = 0 \tag{2.05}$$

We call (2.04) and (2.05) the Leontovich parabolic equations.

In constructing the boundary conditions on the earth's surface (h = 0) we can neglect the difference between the dielectric constant in air and unity.

On the other hand, we can improve these conditions somewhat by using our results which were obtained by the series summation method (see [2] and [4]). This improvement reduces to replacing η by $\eta + 1$ in (1.07), and replacing η by $\eta - 1$ in (1.08). As a result we obtain

$$\frac{\partial U_2}{\partial h} = - \frac{ik}{\sqrt{\eta + 1}} U_2 \quad (\text{for } h = 0) \tag{2.06}$$

and

$$\frac{\partial W_2}{\partial h} = - ik \sqrt{\eta - 1}\, W_2 \quad (\text{for } h = 0) \tag{2.07}$$

Moreover, we should formulate the requirement that, in the region near the source where the curvature of the earth's surface and of the rays can be neglected, there should be a reflecting formula for the earth plane. If the height of the source above the earth is $h' = b - a$ then this requirement means that in the aforementioned region there should be:

$$U_2 = \sqrt{\frac{a}{s}} \left\{ e^{i \frac{k(h-h')^2}{2s}} + e^{i \frac{k(h+h')^2}{2s}} \cdot \frac{h + h' - \frac{s}{\sqrt{\eta + 1}}}{h + h' + \frac{s}{\sqrt{\eta + 1}}} \right\} \tag{2.08}$$

and

$$W_2 = \sqrt{\frac{a}{s}} \left\{ e^{i \frac{k(h-h')^2}{2s}} + e^{i \frac{k(h+h')^2}{2s}} \cdot \frac{h + h' - s\sqrt{\eta - 1}}{h + h' + s\sqrt{\eta - 1}} \right\} \tag{2.09}$$

The factors multiplying the second exponentials are the approximate values of the Fresnel coefficients for vertical and horizontal polarization. These last two formulas are generalizations of our formula (1.28) of [1].

Let us note that the expressions (2.08) and (2.09) satisfy approximately the boundary conditions (2.06) and (2.07).

In the case of a field above a perfectly conducting surface ($\eta = \infty$) the boundary conditions (2.06) and (2.07) become

$$\frac{\partial U_2}{\partial h} = 0 \qquad (\text{for } h = 0) \tag{2.10}$$

and

$$W_2 = 0 \qquad (\text{for } h = 0) \tag{2.11}$$

and the reflection formulas are written as

$$U_2 = \sqrt{\frac{a}{s}} \left\{ \exp\left[ik \frac{(h-h')^2}{2s}\right] + \exp\left[ik \frac{(h+h')^2}{2s}\right] \right\} \tag{2.12}$$

and

$$W_2 = \sqrt{\frac{a}{s}} \left\{ \exp\left[ik \frac{(h-h')^2}{2s}\right] - \exp\left[ik \frac{(h+h')^2}{2s}\right] \right\} \tag{2.13}$$

Section 3. Analogy With the Unsteady Problem of Quantum Mechanics

The problem, formulated in the preceding paragraph, of wave propagation in a spherical layer with variable refraction index is analogous to the quantum-mechanical problem of the motion of a wave packet in a given force field.

Let us write Schroedinger's equation for the motion of a particle of mass m_o in a force field with the potential energy Φ. Denoting the particle coordinate by x, the time by t, Planck's constant (divided by 2π) by $\hbar$ we will have:

$$\frac{\partial^2 \psi}{\partial x^2} + 2i \frac{m_o}{\hbar} \frac{\partial \psi}{\partial t} - \frac{2m_o}{\hbar^2} \Phi \psi = 0 \tag{3.01}$$

Comparing Schroedinger's equation (3.01) with the Leontovich equation (2.04) or (2.05) for U_2 and W_2 we see that these equations have identical form with the coordinate x proportional to the height h and the time t proportional to the horizontal distance s and the potential energy Φ proportional to the negative of $\epsilon - 1 + \frac{2h}{a}$ which differs from the so-called reduced (or modified) refraction index

$$M = 10^6 \left(\frac{\epsilon - 1}{2} + \frac{h}{a}\right) = 10^6 \left(n - 1 + \frac{h}{a}\right) \qquad (3.02)$$

only by a constant factor.

Therefore, the Leontovich parabolic equation for the amplitude of the steady process coincides with the unsteady form of the Schroedinger equation.

The resemblance between the two problems is not limited to the agreement of the differential equations but extends to the boundary and "initial" conditions.

There corresponds to the case, considered in quantum mechanics, of the self-conjugate differential equations and boundary conditions, the problem in electromagnetics, of the absence of absorption in air and on the earth, i.e., the case when the refraction index of air is real and the earth is a perfect conductor. This case is most interesting for the superrefraction problem. Besides, the quantum-mechanical methods can be generalized to the case when absorption is present.

If the earth is a perfect conductor, then the boundary conditions for U_2 and W_2 become (2.10) and (2.11) and the conditions of the quantum-mechanical problem corresponding to them are:

$$\frac{\partial \psi}{\partial x} = 0 \qquad \text{(for } x = 0\text{)} \qquad (3.03)$$

or

$$\psi = 0 \qquad \text{(for } x = 0\text{)} \qquad (3.04)$$

As regards the initial conditions, their general form consists in assigning the initial value of the wave function

$$\psi = \psi_o(x) \qquad (\text{for } t = 0 \text{ , } 0 < x < \infty) \qquad (3.05)$$

The function ψ which satisfies the differential equation, the initial and boundary conditions can be sought in the form

$$\psi(x,t) = \int_0^\infty F(x,x',t)\ \psi_o(x')\ dx' \qquad (3.06)$$

For all x' the function F should satisfy the differential equation

$$\frac{\partial^2 F}{\partial x^2} + 2i\frac{m_o}{h}\frac{\partial F}{\partial t} - \frac{2m_o}{h^2}\Phi F = 0 \qquad (3.07)$$

and boundary conditions of the form of (3.03) or (3.04) (the same as ψ). In order that (3.06) should reduce to $\psi_o(x)$ at $t = 0$, F must, as $t \longrightarrow 0$ have a singularity, the character of which is related to the boundary conditions. In the case of the condition

$$\frac{\partial F}{\partial x} = 0 \qquad (\text{for } x = 0) \qquad (3.08)$$

the singularity of F must have the form

$$F(x,x',t) = e^{-\frac{i\pi}{4}}\sqrt{\frac{m_o}{2\pi h t}}\left(\exp\left[\frac{im_o(x-x')^2}{2ht}\right] + \exp\left[\frac{im_o(x+x')^2}{2ht}\right]\right)$$

(3.09)

In the case of the condition

$$F = 0 \qquad (\text{for } x = 0) \tag{3.10}$$

the singularity of F should have the form

$$F(x,x',t) = e^{-\frac{i\pi}{4}}\sqrt{\frac{m_o}{2\pi\hbar t}}\left(\exp\left[\frac{im_o(x-x')^2}{2\hbar t}\right] - \exp\left[\frac{im_o(x+x')^2}{2\hbar t}\right]\right) \tag{3.11}$$

Comparing these formulas with (2.12) and (2.13) we see that for corresponding boundary conditions, the singularity of F agrees exactly with the singularity of U_2 and W_2. Actually, equating the height h to the coordinate x we should put

$$h = x \; ; \quad h' = x' \; ; \quad \frac{s}{k} = \frac{\hbar t}{m_o} \tag{3.12}$$

Expressing F through the variables h, h' and s we will have for the boundary condition (3.08)

$$F = F_2(h,h',s) \tag{3.13}$$

where F_2 satisfies the same equation as U_2, the boundary condition

$$\frac{\partial F_2}{\partial h} = 0 \qquad (\text{for } h = 0) \tag{3.14}$$

and has the singularity

$$F_2(h,h',s) = e^{\frac{-i\pi}{4}}\sqrt{\frac{k}{2\pi s}}\left(\exp\left[\frac{ik(h-h')^2}{2s}\right] + \exp\left[\frac{ik(h+h')^2}{2s}\right]\right) \tag{3.15}$$

For the boundary condition (3.10) we put

$$F = G_2(h,h',s) \tag{3.16}$$

where G_2 satisfies the same differential equation as W_2, the boundary condition

$$G_2 = 0 \quad , \text{(for } h = 0) \tag{3.17}$$

and has the singularity

$$G_2(h,h',s) = e^{\frac{-i\pi}{4}} \sqrt{\frac{k}{2\pi s}} \left(\exp\left[\frac{ik(h-h')^2}{2s}\right] - \exp\left[\frac{ik(h+h')^2}{2s}\right] \right) \tag{3.18}$$

We see that F_2 differs from U_2 only by a constant factor, as does G_2 from W_2, and we have exactly

$$U_2 = e^{\frac{i\pi}{4}} \sqrt{\frac{k}{2\pi a}} \; F_2 \tag{3.19}$$

and

$$W_2 = e^{\frac{i\pi}{4}} \sqrt{\frac{k}{2\pi a}} \; G_2 \tag{3.20}$$

If we denote by $f(h,s)$ the function which satisfies the same equation and the same boundary conditions as U_2 and takes, for $s = 0$, the value

$$f(h,s) = f_o(h) \qquad \text{(for } s = 0) \tag{3.21}$$

then we can write, on the basis of (3.19)

$$f(h,s) = e^{\frac{-i\pi}{4}} \sqrt{\frac{k}{2\pi a}} \int_0^\infty U_2(h,h',s) \; f_o(h') \; dh' \tag{3.22}$$

Similarly, if $f(h,s)$ satisfies the same boundary conditions as W_2 then

$$f(h,s) = e^{\frac{-i\pi}{4}} \sqrt{\frac{k}{2\pi a}} \int_0^\infty W_2(h,h',s)\, f_o(h')\, dh' \qquad (3.23)$$

The last two formulas are correct not only for boundary conditions corresponding to a perfectly conducting earth (when there is an analogy with quantum mechanics) but even for the more general boundary conditions (2.06) and (2.07) where the singularities of U_2 and W_2 are then given by (2.08) and (2.09).

If the function $f_o(h)$ is not zero only in the neighborhood of the point h = h', where the integral of f_o over this region is finite, then f(h,z) will be proportional to U_2 or W_2, respectively, for not too small s. Therefore, U_2 and W_2 correspond to a point source at the height h', as it should be.

In quantum-mechanical language, it can be said that the function ψ, proportional to U_2 or W_2, is the solution of the problem of the dispersion of a wave packet originally concentrated in the neighborhood of one point.

From quantum-mechanics, it is known that the speed of dispersion depends, essentially, on the form of the potential energy. Let us imagine that the particle motion is bounded on one side by an impermeable wall. If the potential energy is such that the force is always directed out of the wall, then dispersion takes place rapidly. If the force holds the particles in some region where the potential energy has a minimum, or near the wall, then dispersion takes place slowly or not at all. In this case the Schroedinger equation admits a solution corresponding to the steady or almost-steady state.

At the initial instant, the wave function of the almost-steady state is not zero only in the region of minimum potential energy. In the course of time, the amplitude of the wave function in this region decreases, and disintegration of the initial almost-steady state takes place. The decrease in the amplitude occurs exponentially and the rapidity of disintegration is characterized by the coefficient in the exponent, which is called the disintegration constant.

If the initial wave function itself is not a wave function of the almost-steady state, the term corresponding to the almost-steady state can be separated out in its expansion and for large values of time this will be the principal term.

In our electromagnetic problem, the horizontal distance s acts the part of the time t of the quantum mechanics problem. The decrease in the amplitude of the field with increasing horizontal distance corresponds to the dispersion of the wave packet, and the earth's surface (h = 0) acts like the wall. The wall will be impermeable if the earth is an absolute conductor; for finite conductivity, the wall will be absorbent and a decrease of the amplitude will take place not only at the expense of waves escaping into the upper layers of the atmosphere but at the expense of the earth absorbing it. As we saw, the part of the potential energy is played by the reduced refraction index, M, taken with the opposite sign. The behavior of the reduced refraction index depending on height is shown in Fig. 1.

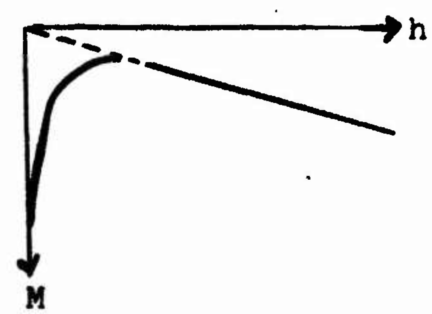

Figure 1. Dependence of the Reduced Refraction Index on Height

The solid curve is the behavior of M with superrefraction. The dotted continuation of the rectilinear part of the curve corresponds to the case when there is no superrefraction and the "equivalent radius" of the earth can be introduced, which is proportional to the angular coefficient of the line relative to the M axis.

If the curve in Fig. 1 is considered as the potential energy curve, then it will be clear that the presence of the maximum for (-M) (minimum for M) which is characteristic for superrefraction, is a necessary condition for the existence of an almost-steady state. Actually, if we denote by h_m the height corresponding to maximum potential energy, then the force in the region $h < h_m$ will be as though squeezing the wave packet to the wall and not letting it go into the $h > h_m$ region.

But in our electromagnetic problem, the presence of an almost-steady state denotes such wave propagation in which its amplitude decreases with increasing distance abnormally slowly, so that its attenuation coefficient (corresponding to constant disintegration) is abnormally small. Hence it follows that the existence condition of the almost-steady state is a condition of the possibility of extra-far propagation of radio waves.

The analogy with quantum mechanics, which was carried out here, permits formation of a qualitative picture of the phenomena of extra-far radio wave propagation. This analogy is useful, so that, certain mathematical methods applicable in quantum mechanics can be transferred into the radiophysics domain. On the other hand, the methods, developed by us, of solving the radiowave propagation problems can be applied in quantum mechanics. However, this question is beyond the scope of this paper.

Section 4. Transformation to Nondimensional Quantities

Let us return to the solution of the problem formulated in Sec. 2. It is necessary to determine U_2 and W_2 which satisfy the differential equations (2.04) and (2.05), the boundary conditions (2.06) and (2.07) and the conditions (2.08) and (2.09) characterizing the singularity. This problem was solved, earlier, for two cases: a) inhomogeneous atmosphere, source on the earth, and b) homogeneous atmosphere, raised source. Now we show that this problem can be solved for the general case of the inhomogeneous atmosphere and the raised source.

Let us transform, in our equations, to the nondimensional quantities used in our previous work. To do this, let us consider the coefficient of U_2 in (2.04). This coefficient is proportional to the quantity

$$\frac{\epsilon - 1}{2} + \frac{h}{a} = 10^{-6} M(h) \qquad (4.01)$$

where $M(h)$ is the "modified" refraction index. We assume that, starting with some height $h = H$, this quantity can be approximated by a linear function of h and we put

$$\frac{\epsilon - 1}{2} + \frac{h}{a} = \alpha + \frac{h}{a^*} \qquad (4.02)$$

where a^* is the so-called equivalent radius of the earth and α is some small constant (for example, $\alpha < 0.0005$). In the simplest case, it is possible to consider that $\epsilon = 1$ for $h > H$; then it is necessary to put $\alpha = 0$ and $a^* = a$.

In that region where (4.02) is correct, the equation for U_2 becomes:

$$\frac{\partial^2 U_2}{\partial h^2} = 2ik \frac{\partial U_2}{\partial s} + k^2 \left(2\alpha + \frac{2h}{a^*}\right) U_2 = 0 \qquad (4.03)$$

In order to get rid of the α in the last term, let us make the substitution

$$U_2 = C\, e^{i\alpha ks}\, \Psi \tag{4.04}$$

where C is a constant which we dispose of later. Then (4.03) is reduced to

$$\frac{\partial^2 \Psi}{\partial h^2} + 2ik \frac{\partial \Psi}{\partial s} + k^2 \frac{2h}{a^*} \Psi = 0 \tag{4.05}$$

Let us introduce the abbreviation

$$m = \left(\frac{ka^*}{2}\right)^{\frac{1}{3}} \tag{4.06}$$

and let us put

$$ks = 2m^2 x \; ; \quad kh = my \; ; \quad kh' = my' \tag{4.07}$$

Then (4.05) can be written

$$\frac{\partial^2 \Psi}{\partial y^2} + i \frac{\partial \Psi}{\partial x} + y\, \Psi = 0 \tag{4.08}$$

The same substitutions reduce the more exact equation (2.04) to the form:

$$\frac{\partial^2 \Psi}{\partial y^2} + i \frac{\partial \Psi}{\partial x} + \left[y + r(y)\right] \Psi = 0 \tag{4.09}$$

where

$$r(y) = m^2 \left(\epsilon - 1 + \frac{2h}{a} - 2\alpha - \frac{2h}{a^*}\right) \tag{4.10}$$

The quantity $r(y)$ characterizes the anomalous behavior of the refraction index near the earth's surface. Starting with some value y, $r(y)$ can be set equal to zero. If we consider that $\alpha = 0$ and $a^* = a$, then simply,

$$r(y) = m^2(\epsilon - 1) \tag{4.11}$$

There remains to express the boundary conditions and the conditions characterizing the singularity in the new variables. Setting

$$q = \frac{im}{\sqrt{\eta + 1}} \tag{4.12}$$

we will have

$$\frac{\partial \Psi}{\partial y} + q\,\Psi = 0 \qquad (\text{for } y = 0) \tag{4.13}$$

We chose the constant C in (4.04) so that the equation analogous to (3.22) can be written in the form

$$f(x,y) = \int_0^\infty \Psi(x,y,y')\, f_o(y')\, dy' \tag{4.14}$$

Then the equation defining the singularity of Ψ, becomes

$$\Psi = \frac{e^{\frac{-i\pi}{4}}}{2\sqrt{\pi x}} \left\{ \exp\left[\frac{i(y-y')^2}{4x}\right] + \exp\left[\frac{i(y+y')^2}{4x}\right] \frac{y + y' + 2iqx}{y + y' - 2iqx} \right\} \tag{4.15}$$

From the comparison of (2.08) with (4.15) we obtain

$$C = \frac{\sqrt{2\pi ka}}{m} \exp\left(\frac{i\pi}{4}\right) \tag{4.16}$$

The function W_2 differs from U_2 only in that the quantity $\frac{1}{\sqrt{\eta + 1}}$ in the boundary conditions and in the equation defining the singularity is replaced by $\sqrt{\eta - 1}$. This corresponds to replacing q by

$$q_1 = im\sqrt{\eta - 1} \tag{4.17}$$

In practice, it is possible to put $q_1 = \infty$ in all cases.

Along with Ψ we will consider the function

$$V(x,y,y'q) = 2\sqrt{\pi x}\ \exp\left[\frac{i\pi}{4}\right]\Psi \tag{4.18}$$

which we will call the attenuation factor. The quantities U_2 and U are expressed through V as follows:

$$U_2 = e^{i\alpha ks}\sqrt{\frac{a}{s}}\ V \tag{4.19}$$

and

$$U = \frac{e^{i(1+\alpha)ks}}{\sqrt{sa\sin\frac{s}{a}}}\ V(x,y,y',q) \tag{4.20}$$

The function W is obtained from (4.20) by replacing q by q_1.

Section 5. Properties of Particular Solutions of the Differential Equations

In order to construct the function Ψ satisfying the formulated conditions, it is necessary to investigate the properties of particular solutions of (4.09) which are obtained by separation of variables. Putting

$$\Psi = e^{ixt}\ f(y,t) \tag{5.01}$$

we obtain for f(y,t)

$$\frac{\partial^2 f}{\partial y^2} + \left[y + r(y) - t\right] f = 0 \tag{5.02}$$

Let us denote the solutions of (5.02) through $f^{o}(y,t)$ and $f^{*}(y,t)$, which satisfy the initial conditions:

$$f^{o}(0,t) = 1 \;; \quad \left(\frac{\partial f^{o}}{\partial y}\right)_{y=0} = 0 \qquad (5.03)$$

and

$$f^{*}(0,t) = 1 \;; \quad \left(\frac{\partial f^{*}}{\partial y}\right)_{y=0} = 1 \qquad (5.04)$$

The general solution of (5.02) will have the form

$$f(y,t) = A^{o}f^{o}(y,t) + A^{*}f^{*}(y,t) \qquad (5.05)$$

On the other hand, if $r(y)$ decreases sufficiently rapidly as y increases, then for real t (5.02) will have one integral (determined to the accuracy of a constant factor independently of y) which will act as $w_1(t - y)$ for large y, and another integral which will behave as $w_2(t - y)$ where w_1 and w_2 are complex Airy functions which represent the solutions of the equation

$$\frac{d^2w}{dy^2} + (y - t)\, w = 0 \qquad (5.06)$$

obtained from (5.02) by replacing $r(y)$ by zero. The functions w_1 and w_2 have the asymptotic expressions

$$w_1(t - y) = e^{\frac{i\pi}{4}} (y - t)^{-\frac{1}{4}} e^{i\frac{2}{3}(y - t)^{3/2}} \qquad (5.07)$$

and

$$w_2(t - y) = e^{\frac{-i\pi}{4}} (y - t)^{-\frac{1}{4}} e^{-i\frac{2}{3}(y - t)^{3/2}} \qquad (5.08)$$

Consequently, the behavior of the general integral of (5.02) as $y \longrightarrow \infty$ for real t can be characterized by the constants C_1 and C_2 in the expression:

$$f(y,t) = C_1 w_1(t - y) + C_2 w_2(t - y) \tag{5.09}$$

Let us establish the relation between the constants A^o, A^*, C_1, C_2 (which can be functions of the parameter t).

By virtue of (5.02) and (5.06), we have:

$$\frac{d}{dy}\left(w \frac{df}{dy} - f \frac{dw}{dy}\right) = - r(y) \cdot f \cdot w(t - y) \tag{5.10}$$

In this equality, we can put successively $w = w_1$ then $w = w_2$ and then integrate between 0 and ∞. As a consequence of the relation

$$\frac{\partial w_1}{\partial y} w_2 - \frac{\partial w_2}{\partial y} w_1 = 2i \tag{5.11}$$

we will have

$$\lim_{y \to \infty} \left(w_2 \frac{df}{dy} - f \frac{dw_2}{dy}\right) = 2iC_1 \tag{5.12}$$

and

$$\lim_{y \to \infty} \left(w_1 \frac{df}{dy} - f \frac{dw_1}{dy}\right) = - 2iC_2 \tag{5.13}$$

and, after integration, (5.10) yields:

$$2iC_1 = A^o w_2'(t) + A^* w_2(t) - \int_0^\infty r(y)\, f(y,t)\, w_2(t - y)\, dy \tag{5.14}$$

and

$$- 2iC_2 = A^{o}w_1'(t) + A^{*}w_1(t) - \int_0^{\infty} r(y)\ f(y,t)\ w_1(t - y)\ dy \tag{5.15}$$

If, here, we substitute (5.05) in place of $f(y,t)$ we obtain the desired relation between the constants A^o, A^*, C_1, C_2 in the form:

$$2iC_1 = A^{o}\left\{w_2'(t) - \int_0^{\infty} r(y)\ f^{o}(y,t)\ w_2(t - y)\ dy\right\} + A^{*}\left\{w_2(t) - \int_0^{\infty} r(y)\ f^{*}(y,t)\ w_2(t - y)\ dy\right\} \tag{5.16}$$

and

$$-2iC_2 = A^{o}\left\{w_1'(t) - \int_0^{\infty} r(y)\ f^{o}(y,t)\ w_1(t - y)\ dy\right\} + A^{*}\left\{w_1(t) - \int_0^{\infty} r(y)\ f^{*}(y)\ w_1(t - y)\ dy\right\} \tag{5.17}$$

Let us observe that the coefficients of the A^o and A^* in these equations are integral, transcendental functions of t. Actually, f^o, f^*, w_1, w_2 are integral functions of t; the integration can be carried out, in practice, between finite limits since $r(y)$ can be set equal to zero starting with some y. (The same conclusion will be correct and without this limitation on r(y) if only r(y) decreases sufficiently rapidly at infinity.)

Hence it follows that if the constants A^o and A^* will be integral transcendental functions of t, then the constants C_1 and C_2 will have the same character. This allows us to apply (5.16) and (5.17), derived for real t, in the case of arbitrary complex values of t also.

If we put

$$A^o = A_1^o(t) \equiv w_1(t) - \int_0^\infty r(y)\, f^*(y,t)\, w_1(t - y)\, dy \qquad (5.18)$$

and

$$A = A_1(t) \equiv - w_1'(t) + \int_0^\infty r(y)\, f^o(y,t)\, w_1(t - y)\, dy \qquad (5.19)$$

then

$$f_1(y,t) = A_1^o(t)\, f^o(y,t) + A_1^*(t)\, f^*(y,t) \qquad (5.20)$$

will be that solution of (5.02) which behaves as $w_1(t - y)$ as $y \longrightarrow \infty$ and which is at the same time an integral transcendental function of t.

Similarly, if we put

$$A^o = A_2^o(t) = w_2(t) - \int_0^\infty r(y)\, f^*(y,t)\, w_2(t - y)\, dy \qquad (5.21)$$

and

$$A^{*} = A_2^{*}(t) = - w_2'(t) - \int_0^{\infty} r(y)\, f^{o}(y,t)\, w_2(t - y)\, dy \qquad (5.22)$$

then

$$f_2(y,t) = A_2^{o}(t)\, f^{o}(y,t) + A_2^{*}(t)\, f^{*}(y,t) \qquad (5.23)$$

will behave as $w_2(t - y)$ as $y \longrightarrow \infty$ and will be an integral function of t.

The integral $f_1(y,t)$ will have the asymptotic expression

$$f_1(y,t) = \frac{c' e^{\frac{i\pi}{4}}}{\sqrt[4]{y - t + r(y)}} \exp\left[i \int_{\tau}^{y} \sqrt{u - t + r(u)}\, du \right] \qquad (5.24)$$

and the integral $f_2(y,t)$ will have the asymptotic expression

$$f_2(y,t) = \frac{c'' e^{\frac{-i\pi}{4}}}{\sqrt[4]{y - t + r(y)}} \exp\left[-i \int^{y} \sqrt{u - t + r(u)}\, du \right] \qquad (5.25)$$

where c', c" and τ are constants. If we put $r(y) = 0$, $\tau = t$, $c' = c'' = 1$ then (5.24) and (5.25) will transform into the asymptotic expressions (5.07) and (5.08) for w_1 and w_2.

We already used the integral $f_1(y,t)$ in [1] where, however, it was assumed without proof that such an integral exists, which has the asymptotic expression (5.24) and is, meanwhile, an integral transcendental function of t. The proof of this statement, which is reproduced here, can be used also for practical (numerical) construction of this integral.

For complex t the function $f_1(y,t)$ will increase as y increases and the integral of the square of $f_1(y,t)$, taken over y from 0 to ∞ will diverge. However, for certain assumptions on the behavior of $r(y)$ in the complex plane, $f_1(y,t)$ will behave as $w_1(t-y)$ for complex y and will converge to zero on the ray $y = re^{i\alpha}$ (where $\alpha = \frac{\pi}{3}$), so that the integral

$$I = \int_0^{\infty e^{i\alpha}} f_1^2(y,t)\,dy \tag{5.26}$$

will converge. Let us evaluate this integral. Differentiating (5.02) with respect to t, we obtain

$$\frac{d}{dy^2}\left(\frac{\partial f}{\partial t}\right) + \left[y + r(y) - t\right]\frac{\partial f}{\partial t} = f \tag{5.27}$$

Hence, from (5.02) we obtain the relation

$$\int_0^y f^2\,dy = \left(f\,\frac{\partial^2 f}{\partial y \partial t} - \frac{\partial f}{\partial y}\,\frac{\partial f}{\partial t}\right)\Bigg|_0^y \tag{5.28}$$

Putting, here, $f = f_1(y,t)$ and considering the upper limit of integration to be equal to $\infty e^{i\alpha}$ we will have:

$$\int_0^{\infty e^{i\alpha}} f_1^2(y,t)\,dy = -\left(f_1\,\frac{\partial^2 f_1}{\partial y \partial t} - \frac{\partial f_1}{\partial t}\,\frac{\partial f_1}{\partial y}\right)_0 \tag{5.29}$$

Section 6. Construction of the Solution as a Contour Integral or Series

In the previous paragraph we established the existence of two integrals of the ordinary differential equation

$$\frac{d^2f}{dy^2} + \left[y + r(y) - t\right] f = 0 \tag{6.01}$$

which are integral transcendental functions of the parameter t and have the asymptotic expansions (5.24) and (5.25). These integrals, which we denoted by $f_1(y,t)$ and $f_2(y,t)$, are determined by (5.20) and (5.23).

We show now that with the aid of f_1 and f_2 we can construct contour integrals for V and Ψ which are the solutions of our problem. Our reasoning will be similar to the reasoning explained in Sec. 3 of [1], and the final formulas will be analogs of (2.24) and (3.10) of [2].

Let us denote the Wronskian by $D_{12}(t)$:

$$D_{12}(t) = f_1 \frac{\partial f_2}{\partial y} - f_2 \frac{\partial f_1}{\partial y} \tag{6.02}$$

and let us put

$$F(t,y,y',q) = \frac{1}{D_{12}(t)} \; f_1(y',t) \left\{ f_2(y,t) - \frac{f_2'(0,t) + qf_2(0,t)}{f_1'(o,t) + qf_1(0,t)} \; f_1(y,t) \right\} \tag{6.03}$$

where the primes of the f_1 and f_2 denote derivatives with respect to y. Let us consider $y' > y$, and let us form the integral

$$\Psi = \frac{1}{2\pi i} \int e^{ixt} \; F(t,y,y',q) \, dt \tag{6.04}$$

taken around a contour which envelops all the poles of the integrand in a positive d. tion.

From the definition of F it follows that F is meromorphic in t (i.e. for finite t has no singularities except poles). The function F is completely determined even if the functions f_1 and f_2, which are part of it, are determined only to the accuracy of a factor independent of y. Since, for all values of t the integrals f_1 and f_2 are independent (this is seen from their asymptotic expressions), then the Wronskian $D_{12}(t)$ has no roots and the unique poles of F are the roots of the equation

$$f_1'(0,t) + qf_1(0,t) = 0 \qquad (6.05)$$

If r(y) = 0 in (6.01) then we can put

$$f_1(y,t) = w_1(t - y) \;;\quad f_2(y,t) = w_2(t - y) \qquad (6.06)$$

Then

$$D_{12}(t) = - 2i \qquad (6.07)$$

and the expression (6.03) for F reduces to the formula (2.21) considered in our work [2].

Let us show that Ψ, defined by the contour integral (6.04), satisfies all the conditions.

First of all, it is evident that Ψ satisfies the differential equation (4.09) because the integrand satisfies it. Furthermore, Ψ satisfies the boundary condition (4.13) since we have for all t and y'

$$\frac{\partial F}{\partial y} + q\,F = 0 \qquad (\text{for } y = 0) \qquad (6.08)$$

There remains to show that Ψ has the necessary singularity.

With the aid of the asymptotic expressions (5.24) and (5.25) we can show that if x,y are small and the ratio y/x is large then the principal part of the integration in (6.04) will lie at large negative values of t. But if t is large and negative, then the term $-t$ will play the main part in the coefficient of f in the differential equation (6.01). Consequently, for large negative t we will have, approximately:

$$f_1(y,t) \sim f_1(0,t)\ e^{iy\sqrt{-t}} \tag{6.09}$$

and

$$f_2(y,t) \sim f_2(0,t)\ e^{-iy\sqrt{-t}} \tag{6.10}$$

Substituting these expressions in (6.03) for F, we obtain

$$F = \frac{1}{2\sqrt{-t}} \left\{ \exp\left[i(y'-y)\sqrt{-t}\right] - \frac{q - i\sqrt{-t}}{q + i\sqrt{-t}} \exp\left[i(y'+y)\sqrt{-t}\right] \right\} \tag{6.11}$$

Substitution of this value of F in the integral (6.04) yields the Weyl-van der Pol formula for Ψ, which after neglecting small quantities (in the second term) reduces to (4.15) characterizing the singularity of Ψ.

Therefore, the correctness of (6.04) for Ψ is established.

It is not difficult to transform from the contour integral (6.04) into a series, around the residues, referred to the roots of (6.05). Let us write this equation in some detail.

Using (5.20) for $f_1(y,t)$ and the initial values (5.03) and (5.04) of f^o and f^*, we obtain

$$f_1(0,t) = A_1^o(t)\ ; \quad f_1'(0,t) = A_1^*(t) \tag{6.12}$$

and (6.05) becomes

$$A_1^*(t) + q\ A_1^o(t) = 0 \tag{6.13}$$

Substituting here the values (5.18) and (5.19) of A_1^o and A_1^* we will have:

$$w_1'(t) - qw_1(t) - \int_0^{\infty} r(y) \left[f^o(y,t) - qf^*(y,t)\right] w_1(t-y)\, dy = 0 \tag{6.14}$$

We will call this the characteristic equation.

It is essential, for us, that the left side of the characteristic equation be an integral transcendental function of t and that it contain only the functions $f^o(y,t)$ and $f^*(y,t)$, which can be obtained for all values of t by means of numerical integration of the differential equation (5.02) with the initial conditions (5.03) and (5.04). In that case when r(y), starting with some $y = y_1$, is zero, (6.14) can be integrated and the characteristic equation reduces to

$$w_1'(t-y) \left[f^o(y,t)-qf^*(y,t)\right] + w_1(t-y) \frac{\partial}{\partial y} \left[f^o(y,t) - qf^*(y,t)\right] = 0$$

$$(\text{for } y = y_1) \tag{6.15}$$

The characteristic equation for the case of a homogeneous atmosphere is

$$w_1'(t) - qw_1(t) = 0 \tag{6.16}$$

This equation is obtained from the previous formula by putting $r(y) = 0$ in (6.14) or by putting $y = y_1 = 0$ in (6.15).

We denote the roots of the characteristic equation by

$$t_1(q),\ t_2(q),\ \ldots \tag{6.17}$$

These roots will be functions of the parameter q.

Let us compute the residues (6.04) numerically. From (6.02) and (6.05) for $y = 0$ and $t = t_s$ results

$$\frac{f_2'(0,t) + qf_2(0,t)}{D_{12}(t)} = \frac{1}{f_1(0,t)} \tag{6.18}$$

Moreover, the derivative with respect to t of the denominator in (6.18) is

$$\frac{\partial^2 f_1}{\partial y \partial t} + q \frac{\partial f_1}{\partial t} = f_1 \frac{\partial}{\partial t}\left(\frac{1}{f_1} \frac{\partial f_1}{\partial y}\right) = - f_1(0,t) \frac{dq}{dt} \tag{6.19}$$

Consequently, the residue of F at $t = t_s$ will be

$$\frac{dt_s}{dq} \frac{f_1(y',t_s)}{f_1(0,t_s)} \frac{f_1(y,t_s)}{f_1(0,t_s)} \tag{6.20}$$

Taking the sum of expressions (6.20), multiplying by e^{ixt_s}, we obtain the desired expansion of Ψ in the series

$$\Psi = \sum_{s=1}^{\infty} e^{ixt_s} \frac{dt_s}{dq} \frac{f_1(y',t_s)}{f_1(0,t_s)} \frac{f_1(y,t_s)}{f_1(0,t_s)} \tag{6.21}$$

The quantities

$$\frac{f_1(y,t_s)}{f_1(0,t_s)} = f^0(y,t_s) - qf^*(y,t_s) \tag{6.22}$$

can be called the height factors. Let us note that the height factors are expressed, according to (6.22), through the functions f^0 and f^* which are evaluated directly by means of numerical integration of (5.02).

In that case when q is very large or equal to infinity (horizontal polarization, perfect conductor) (6.21) should be transformed by means of termwise multiplication into

$$\frac{q^2\, f_1^2(0,t_s)}{f_1'^2(0,t_s)} = 1 \tag{6.23}$$

The result can be written

$$\Psi = \sum_{s=1}^{\infty} e^{ixt_s} \left(q^2 \frac{dt_s}{dq}\right) \frac{f_1(y',t_s)}{f_1'(0,t_s)} \frac{f_1(y,t_s)}{f_1'(0,t_s)} \tag{6.24}$$

The quantity

$$q^2 \frac{dt_s}{dq} = 1 : \frac{d}{dt}\left(\frac{f_1(0,t)}{f_1'(0,t)}\right) \tag{6.25}$$

will be finite for $q \rightarrow \infty$. Let us note that from (6.19) and (5.29) result

$$f_1^2(0,t) \frac{dq}{dt} = \int_0^{\infty e^{i\alpha}} f_1^2(y,t)\, dy \qquad \left(\alpha = \frac{\pi}{3}\right) \tag{6.26}$$

Consequently, the series (6.21) can be written

$$\Psi = \sum_{s=1}^{\infty} e^{ixt_s} \left\{ \frac{f_1(y',t_s)\, f_1(y,t_s)}{\displaystyle\int_0^{\infty e^{i\alpha}} f_1^2(y,t_s)dy} \right\} \tag{6.27}$$

In such a form, it recalls expansions in terms of eigenfunctions. In (6.27) the "eigenvalues" are, however, complex and the "normalized integrals" in the denominator converge only for complex paths of integration.

In order to transform from Ψ to the attenuation factor V, it is sufficient to recall (4.18)

$$V(x,y,y',q) = 2\sqrt{\pi x}\ \exp\left[\frac{i\pi}{4}\right]\Psi$$

In the case of a homogeneous atmosphere, when it is possible to put $r(y) = 0$ and $f_1(y,t) = w_1(t-y)$ which results from our formula, the expressions for V reduce to just what was derived in our earlier work [2] by another method.

Section 7. Application of the General Theory to the Superrefraction Case (Schematic Example)

The analog, considered in Sec. 3, with the unsteady problems of quantum mechanics permits the formation of a qualitative picture of the superrefraction phenomenon and those conditions for which this phenomenon may occur. On the other hand, the general expression, obtained in Sec. 6, for the attenuation factor is suitable for quantitative computations which, by right, require sufficiently complex calculations.

Let us write the expression for Ψ which is related to the attenuation factor. For brevity, putting

$$f(y,t) = f^{o}(y,t) - qf^{*}(y,t) \qquad (7.01)$$

we will have, on the basis of (6.21)

$$\sum_{s=1}^{\infty} e^{ixt_s}\frac{dt_s}{dq} f(y',t_s)\,f(y,t_s) \qquad (7.02)$$

where the t_s are the roots of the transcendental equation (6.14). If $r(y) = 0$ for $y > y_1$, this equation can be written according to (6.15) as:

$$w_1'(t-y_1)\ f(y_1,t) + w_1(t-y_1)\ f'(y_1,t) = 0 \qquad (7.03)$$

where w_1' denotes the derivative with respect to the argument $(t-y)$ but not the derivative with respect to y. The parameter q enters into this equation by means of (7.01).

The determination of the conditions for which extra-far propagation is possible reduces to the study of the roots of the characteristic equations (6.14) or (7.03). In the absence of superrefraction, the imaginary part of the roots of this equation, which according to (7.02) yields the attenuation of the waves with increasing distance, will be of the same order as the real part. When there is superrefraction, there exists one or more roots with abnormally small imaginary part.

In order to formulate the representation of the conditions under which extra-far propagation can occur, let us consider the following schematic example.

Let the function $r(y)$ be the following

$$\begin{aligned} r(y) &= (1+\mu^3)\,(y_1 - y) \quad && (\text{for } 0 < y < y_1) \\ r(y) &= 0 && (\text{for } y_1 < y) \end{aligned} \qquad (7.04)$$

This corresponds to the assumption that the graph of the refraction index is a broken line as shown in Fig. 2.

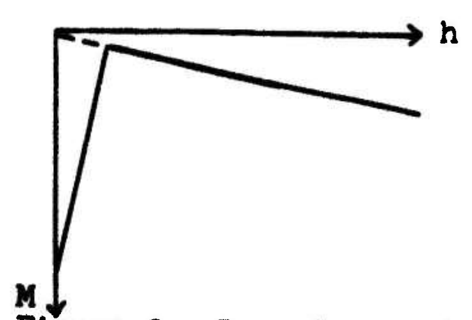

Figure 2. Dependence of the Reduced Refraction Index on Height (Schmatic Ex.)

If the dielectric constant ϵ is considered to vary thus:

$$\begin{aligned} \epsilon &= 1 - g(h - h_1), &\quad &(\text{for } h < h_1) \\ \epsilon &= 1 &\quad &(\text{for } h > h_1) \end{aligned} \tag{7.05}$$

then the parameter μ^3 and y_1 will equal

$$\mu^3 = \frac{ag}{2} - 1 \; ; \quad y_1 = h_1 \sqrt[3]{\frac{2k^2}{a}} \tag{7.06}$$

Therefore the parameter μ depends on the wave length, but the parameter y_1 (the reduced height of the break point) will be proportional to $\lambda^{-2/3}$.

We write the equations for f as

$$\frac{d^2 f}{dy^2} + \left[(1 + \mu^3)\, y_1 - \mu^3 y - t\right] f = 0$$

$$(\text{for } y < y_1)$$

$$\tag{7.07}$$

$$\frac{d^2 f}{dy^2} + (y - t)\, f = 0 \qquad (\text{for } y > y_1)$$

Let us introduce instead of t the new parameter

$$\xi_o = \frac{t - (1 + \mu^3) y_1}{\mu_2} \tag{7.08}$$

and instead of y, the new variable

$$\xi = \xi_o + \mu y \tag{7.09}$$

The value $\xi = \xi_1$ will correspond to $y = y_1$, where

$$\mu^2 \xi_1 = t - y_1 \tag{7.10}$$

Equation (7.05) becomes

$$\frac{d^2f}{d\xi^2} = \xi f \qquad (\xi_o < \xi < \xi_1) \tag{7.11}$$

The Airy functions $u(\xi)$ and $v(\xi)$ will be independent solutions of the latter. The functions f^o and f^* will equal

$$\begin{aligned} f^o(y) &= u'(\xi_o)\, v(\xi) - v'(\xi_o)\, u(\xi) \\ f^*(y) &= \frac{1}{\mu}\left[v(\xi_o)\, u(\xi) - u(\xi_o)\, v(\xi)\right] \end{aligned} \tag{7.12}$$

By virtue of the relation

$$u'(\xi)\, v(\xi) - v'(\xi)\, u(\xi) = 1 \tag{7.13}$$

the functions f^o and f^* will satisfy the initial conditions (5.03) and (5.04). Introducing according to (7.01) the function

$$f(y) = -\frac{1}{\mu}\left[qv(\xi_o) + \mu v'(\xi_o)\right] u(\xi) + \frac{1}{\mu}\left[qu(\xi_o) + \mu u'(\xi)\right] v(\xi) \tag{7.14}$$

we obtain the characteristic equation if we substitute the values of $f(y)$ and $f'(y)$ for $y = y_1$ in (7.03). This characteristic equation can be written as

$$\frac{\mu v'(\xi_o) + qv(\xi_o)}{\mu u'(\xi_o) + qu(\xi_o)} = \frac{\mu v'(\xi_1)w_1(\mu^2\xi_1) + v(\xi_1)w_1'(\mu^2\xi_1)}{\mu u'(\xi_1)w_1(\mu^2\xi_1) + u(\xi_1)w_1'(\mu^2\xi_1)} \tag{7.15}$$

Let us assume that the magnitudes of y_1 and the parameter μ are sufficiently large. This means that the "potential well" on Fig. 2 is sufficiently wide and deep. In such a case, the quantities ξ_1 and $\mu^2\xi_1$ [the arguments of the functions in the right side of (7.15)] will be large. By virtue of the asymptotic expansions

$$u(\xi) = \xi^{-\frac{1}{4}} \exp\left[\frac{2}{3}\xi^{3/2}\right]; \qquad v(\xi) = \frac{1}{2}\xi^{-\frac{1}{4}} \exp\left[-\frac{2}{3}\xi^{3/2}\right] \tag{7.16}$$

the right side of (7.15) will be very small and the characteristic equation reduces, approximately, to:

$$\mu v'(\xi_o) + qv(\xi_o) = 0 \tag{7.17}$$

This case will occur when the gradient of the dielectric constant of air, at a sufficiently high region, will be negative and larger than $\frac{2}{a}$, where a is the earth's radius; then the curvature of the ray will be larger than the earth's curvature and formal computations of the "equivalent radius" are negative.

The wave attenuation with increasing horizontal distance is related to the imaginary part of t and, thus, with the imaginary part of ξ_o; if ξ_o were real there would be no attenuation. Attenuation may occur for two reasons: absorption by the earth and escape through the upper layer of the atmosphere. Absorption in the earth is characterized by the complex parameter q. Equation (7.17) corresponds to that case when the attenuation occurs at the expense of absorption by the earth. If the earth be considered an absolute conductor, it is necessary to put $q = 0$ for horizontal polarization and $q = \infty$ for vertical polarization. For $q = 0$, (7.17) reduces to

$$v'(\xi_o) = 0 \qquad (\text{for } q = 0) \tag{7.18}$$

The roots will be real negative numbers

$$\xi_o = -1.019 \ ; \ -3.248 \ ; \ -4.820 \ ; \ \ldots \tag{7.19}$$

For $q = \infty$, (7.17) becomes

$$v(\xi_o) = 0 \qquad (\text{for } q = \infty) \tag{7.20}$$

and has the roots

$$\xi_o = -2.338 \; ; \quad -4.088 \; ; \quad -5.521 \; : \tag{7.21}$$

Because ξ_o is real in these cases, there is no attenuation.

Equation (7.17) will represent a better approximation to (7.15) if ξ_1 (or its real part) be positive and sufficiently large. Since

$$\xi_1 = \xi_o + \mu y_1$$

this condition will be fulfilled starting with some root ξ_o. Consequently, the number of roots with small imaginary part will be finite.

It is possible to derive an approximate formula for the correction to ξ_o, obtainable by computing the right side of (7.15). Let us denote by ξ_{∞} that root of (7.17) which we will consider as the inaccurate value of ξ_o and by $\Delta\xi_o$ - the increment. This correction is obtained if we substitute the value of ξ_1, in the right side of (7.15), which equals

$$\xi_1 = \xi_{\infty} + \mu y_1$$

The approximate value of the correction is obtained from the equation

$$\Delta\xi_o = \frac{1}{\sqrt{-\xi_{oo}}} \left\{ \frac{1}{16} \, \frac{\mu^3 + 1}{\mu^3 \xi_1^{3/2}} \, \exp\left[-\frac{4}{3}\, \xi_1^{3/2} \right] + \frac{1}{4}\, e^{-2S} \right\} \tag{7.22}$$

where

$$S = \frac{2}{3} (\mu^3 + 1) \xi_1^{3/2} \qquad (7.23)$$

Let us note that the imaginary part of the correction is positive. This corresponds to the fact that leakage in the upper layer increases attenuation.

The applicability condition of these formulas is a sufficiently large value of μy_1. Let us recall that we have, according to (7.06):

$$\mu y_1 = h_1 \sqrt[3]{k^2(g - \frac{2}{a})} \qquad (7.24)$$

where g is the gradient of the dielectric constant with opposite sign, a is the earth's radius, and h_1 is the height of the break point on Fig. 2.

The larger the value of μy_1 the larger the number of the almost-steady states with small attenuation. It can be said, roughly, that the number of such states equals the number of roots, ξ_o, not exceeding the parameter μy_1 (in absolute magnitude).

The concept of the ray reflected from the upper boundary layer and from the earth's surface becomes applicable only when the number of almost-steady states (the number of terms in the series (7.02) with low attenuation) becomes large. Generally, the necessary condition for the applicability of the concepts of geometric optics is the slow convergence of (7.02), when a large number of terms will play a part. If there are one or two terms in it (which can correspond to both almost-steady and attenuation states) then the ray concept is not applicable at all.

Section 8. Approximate Formulas for Terms With Low Attenuation

Using a method similar to that which is applied in quantum mechanics, approximate expressions can be derived for the height factors corresponding to terms with low attenuation, and also estimates can be given for that part of the damping coefficient which corresponds to leakage through the upper layer.

In (6.01) let us put

$$y + r(y) = p(y) \tag{8.01}$$

and let us write this equation thus

$$\frac{d^2f}{dy^2} + \left[p(y) - t\right] f = 0 \tag{8.02}$$

In the superrefraction case, the function p(y), proportional to the reduced refraction index, will have a minimum and will increase on both sides of it; to the left of the minimum the largest value will be p(0) and to the right p(y) will increase as y. If t lies between the least value of p(y) and p(0), then the coefficient of f in (8.02) becomes zero for two values of y which we will denote by y_1 and y_2. In the interval $y_1 < y < y_2$ the quantity $\left[p(y) - t\right]$ will be negative, and outside this interval, positive.

The solution of (8.02) in the interval $y_1 < y < y_2$ can be expressed approximately, through Airy functions. Let us put

$$\int_{y_1}^{y} \sqrt{t - p(y)}\, dy = \frac{2}{3} \xi_1^{3/2} \tag{8.03}$$

and

$$\int_{y}^{y_2} \sqrt{t - p(y)}\ dy = \frac{2}{3}\, \xi_2^{3/2} \tag{8.04}$$

and let us denote by S the sum of these quantities which is independent of y

$$S = \int_{y_1}^{y_2} \sqrt{t - p(y)}\ dy \tag{8.05}$$

We can consider the magnitude of S to be large. With such notation we have approximately:

$$f = \sqrt[4]{\frac{\xi_1}{t - p(y)}}\ \left[A_1 u(\xi_1) + B_1\ v(\xi_1)\right] \tag{8.06}$$

and also

$$f = \sqrt[4]{\frac{\xi_2}{t - p(y)}}\ \left[A_2 u(\xi_2) + B_2 v(\xi_2)\right] \tag{8.07}$$

where

$$\frac{2}{3}\, \xi_1^{3/2} + \frac{2}{3}\, \xi_2^{3/2} = S \tag{8.08}$$

and the constants A_1, B_1, A_2, B_2 are related through

$$A_2 = \frac{1}{2}\, B_1 e^{-S}\ ; \quad B_2 = 2A_1 e^{S} \tag{8.09}$$

which result from comparison of the asymptotic expressions for (8.06) and (8.07) at large values of ξ_1 and ξ_2.

For $y > y_2$ we can determine ξ_2 by means of the equality

$$\int_{y_2}^{y} \sqrt{p(y) - t}\ dy = \frac{2}{3} (-\xi_2)^{3/2} \tag{8.10}$$

and using the previous expression (8.07) for f.

Similarly, for $y < y_1$ we can put, in place of (8.03)

$$\int_{y}^{y_1} \sqrt{p(y) - t}\ dy = \frac{2}{3} (-\xi_1)^{3/2} \tag{8.11}$$

and apply (8.06) for f.

Let us chose the constants A, B such that the function f will be proportional to $f_1(y,t)$. We must put

$$A_2 = C_1 \quad ; \quad B_2 = iC_1 \tag{8.12}$$

and therefore

$$A_1 = \frac{1}{2} C_1 e^{-S} \quad ; \quad B_1 = 2C_1 e^{S} \tag{8.13}$$

Then (8.06) and (8.07) become

$$f_1(y,t) = 2C_1 e^{S} \sqrt[4]{\frac{\xi_1}{t - p(y)}} \left\{ v(\xi_1) + \frac{1}{4} e^{-2S} u(\xi_1) \right\} \tag{8.14}$$

and

$$f_1(y,t) = C_1 \sqrt[4]{\frac{\xi_2}{t - p(y)}} w_1(\xi_2) \tag{8.15}$$

Similarly, the following approximate expressions are obtained for $f_2(y,t)$:

$$f_2(y,t) = 2C_2e^S \sqrt[4]{\frac{\xi_1}{t - p(y)}} \left\{ v(\xi_1) - \frac{1}{4} e^{-2S} u(\xi_1) \right\} \quad (8.16)$$

and

$$f_2(y,t) = C_2 \sqrt[4]{\frac{\xi_2}{t - p(y)}} w_2(\xi_2) \quad (8.17)$$

In this approximation, the Wronskian $D_{12}(t)$ appears to equal

$$D_{12}(t) = - 2iC_1C_2 \quad (8.18)$$

For $y < y_1$ the functions $u(\xi_1)$ and $v(\xi_1)$ will be of one order. As a consequence of the smallness of the factor exp $[-2S]$ the second terms in (8.14) and (8.16) will represent small corrections [generally speaking, less than the error of the whole of expression (8.14) or (8.16)]. Consequently, the functions f_1 and f_2 in the $y < y_1$ region will be almost proportional to each other.

Discarding the small increments, the equation defining t can be written:

$$\left(\frac{d\xi}{dy}\right)_0 v'(\xi_0) + qv(\xi_0) = 0 \quad (8.19)$$

Here ξ_0 and $\left(\frac{d\xi}{dy}\right)_0$ denote the values of ξ_1 and $\frac{d\xi_1}{dy}$ at $y = 0$. This equation is analogous to (7.17).

It yields just that part of the attenuation coefficient of the wave which occurs for absorption by the earth. Since the complex parameter q which characterizes the properties of the ground, is known only very roughly, it is sufficient to take the coefficient $\left(\frac{d\xi}{dy}\right)_0$ in a rough approximation and to put, according to (7.06),

$$\left(\frac{d\xi}{dy}\right)_0 = \mu = \sqrt[3]{\frac{ag}{2} - 1} \tag{8.20}$$

where a is the radius of the earth and g is the gradient of the dielectric constant taken with opposite sign. Then (8.19) will reduce to (7.17) which was studied in the preceding paragraph. The roots ξ_0 of (8.19) will be related to the corresponding values of the parameter t through

$$k \int_0^{h_1^*} \sqrt{-\frac{t}{m^2} + \epsilon - 1 + \frac{2h}{a}}\; dh = \frac{2}{3}(-\xi_0)^{3/2} \tag{8.21}$$

where h_1^* is the lesser of the two values of the height h, h_1^* and h_2^*, for which the radical becomes zero.

If ξ_0 is real, then h_1^* and t are real; if ξ_0 is complex, then evaluation of the integral (8.21) requires analytic continuation of the interpolation formula for ϵ into the complex domain.

A necessary condition of the applicability of the previous formulas is the smallness of the quantity e^{-2S}, where S has the value (8.05). In the customary units, the integral which expresses S, is written

$$S = k \int_{h_1^*}^{h_2^*} \sqrt{\frac{t}{m^2} - \left(\epsilon - 1 + \frac{2h}{a}\right)}\; dh \tag{8.22}$$

Determining t from (8.21), it is necessary to verify that the integral S is sufficiently large for this t.

In the case of an absolute conductor ($q = 0$; $q = \infty$) the approximate values of ξ_0 and t obtained from (8.19) and

(8.21) are real. In this case, it can be said that the approximate value of the imaginary part is the correction to ξ_o.

Putting

$$\xi_o = \xi_o' + i\xi_o'' \tag{8.23}$$

we will have

$$\sqrt{-\xi_o'} - \xi_o'' = \frac{1}{4} e^{-2S} \tag{8.24}$$

We will not dwell on the derivation of this formula.

Since ξ_o'' is a small quantity, then the increment $\Delta\xi_o = i\xi_o''$ will correspond to the increment $\Delta t = it'' = \frac{dt}{d\xi_o}\Delta\xi_o$. But the quantity (8.24) multiplied by i is the increment to the integral (8.21). Consequently, we can determine t'' (the imaginary part of t) from

$$t'' \frac{\partial}{\partial t}\left(k \int_0^{h_1^*} \sqrt{-\frac{t}{m^2} + \epsilon - 1 + \frac{2h}{a}}\; dh\right) = -\frac{1}{4} e^{-2S} \tag{8.25}$$

Since the derivative of the integral is negative, then t'' is positive, which corresponds to attenuation.

The formulas which were obtained permit the derivation, also, of approximate expressions for $\frac{dq}{dt}$. According to (6.19) we have

$$\frac{dq}{dt} = -\frac{\partial^2 \log f_1}{\partial y\, \partial t} \tag{8.26}$$

Substituting, here, the value of f_1 from (8.14) and neglecting small quantities, we obtain

$$\frac{dq}{dt} = \left(\frac{\partial \xi}{\partial y}\right)_o \frac{\partial \xi_o}{\partial t} \left(\frac{v'^2(\xi_o)}{v^2(\xi_o)} - \frac{v''(\xi_o)}{v(\xi_o)}\right) \quad (8.27)$$

Here we can put, in a rough approximation,

$$\left(\frac{d\xi}{dy}\right)_o = \mu \quad ; \quad \frac{\partial \xi_o}{\partial t} = \frac{1}{\mu^2} \quad (8.28)$$

[see formulas (7.08) and (7.09)]. Using the differential equation and the limiting conditions for v, we obtain:

$$\frac{dq}{dt} = \frac{q^2}{\mu^3} - \frac{\xi_o}{\mu} \quad (8.29)$$

The first terms of the series (6.21) which possess low attenuation, will be equal, in our approximation, to

$$\Sigma\, e^{ixt} \frac{v(\xi_1)\, v(\xi_1')}{\left(\frac{q^2}{\mu^3} - \frac{\xi_o}{\mu}\right) v^2(\xi_o)} = \Sigma\, e^{ixt} \frac{\mu v(\xi_1)\, v(\xi_1')}{v'^2(\xi_o) - \xi_o\, v^2(\xi_o)} \quad (8.30)$$

where ξ_1' refers to the reduced height y'.

If ξ_o is so much larger in absolute value, that asymptotic expressions can be used for $v(\xi_o)$ and $v'(\xi_o)$ then the denominator in this formula will equal approximately

$$v'^2(\xi_o) - \xi_o\, v^2(\xi_o) = \sqrt{-\xi_o} \quad (8.31)$$

In conclusion, it is necessary to emphasize that the formulas derived in this paragraph are based on rough approximations and are intended for rough computations. More exact computations should be based on the rigorous theory explained in the previous paragraphs.

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VII. THE FIELD FROM A VERTICAL AND A HORIZONTAL DIPOLE, RAISED SLIGHTLY ABOVE THE EARTH'S SURFACE

V. A. Fock

In the book "Diffraction of Radio Waves Around the Earth's Surface"[1] we developed a general method for the summation of the series representing the field from a dipole on the earth's surface. The shape of the earth is assumed spherical. In that work our method was applied to the case of a vertical dipole, located on the surface. The case of a slightly raised vertical dipole is of no less interest. We propose to analyze this case in the present paper.

1. Vertical Raised Dipole. Solution in Series Form.

We will employ the notation used in ref. 1. Let k be the wave vector in air, η the complex dielectric constant of the earth, and $k_2 = k\eta^{1/2}$ the complex wave vector for the earth. For simplicity we will not consider the atmospheric refraction, and remember only[2] that calculation of the refraction effect can be accomplished approximately, if we replace the earth's geometric radius a by an equivalent radius a*.

We introduce spherical coordinates r, θ, ϕ referred to the origin at the center of the earth and with the dipole along the polar axis. The field components in air are expressed in terms of the Hertz function U by the equations:

$$E_r = \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial U}{\partial\theta} \right) \tag{1.1}$$

$$E_\theta = -\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial\theta} \right) \tag{1.2}$$

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$$H_\phi = -ik \frac{\partial U}{\partial\theta} \tag{1.3}$$

1 P. 206

Let the elevation of the dipole above the earth's surface be denoted by $h=b-a$ (so that b is the distance of the dipole from the center of the earth). We introduce the functions $\psi_n(x)$ and $\zeta_n(x)$, related to the Bessel and Hankel functions as follows:

$$\psi_n(x) = \sqrt{\frac{\pi x}{2}}\; J_{n+\frac{1}{2}}(x) \tag{1.4}$$

$$\zeta_n(x) = \sqrt{\frac{\pi x}{2}}\; H^{(1)}_{n+\frac{1}{2}}(x) \tag{1.5}$$

and we denote by $\chi_n(x)$ the logarithmic derivative

$$\chi_n(x) = \psi_n'(x) \,/\, \psi_n(x) \tag{1.6}$$

and by $P_n(\cos\theta)$, the Legendre polynomial.

The expansion of the Hertz function U in the range $a \leq r < b$ will then have the form:

$$U = \frac{1}{kbr} \sum_{n=0}^{\infty} (2n+1)\, \zeta_n(kb) \left[\psi_n(kr) - A_n\, \zeta_n(kr)\right] P_n(\cos\theta) \tag{1.7}$$

where

$$A_n = \frac{k_2\psi_n'(ka) - k\psi_n(ka)\, \chi_n(k_2a)}{k_2\zeta_n'(ka) - k\zeta_n(ka)\, \chi_n(k_2a)} \tag{1.8}$$

These equations are listed on p. 5 of ref. 1. Further calculation there, however, is carried out only for the case $r = a$. We shall now free ourselves from this limitation.

_. Approximate Series Summation for the Hertz Function

For the approximate summation of the series we can apply unchanged the method outlined in ref. 1. We write the series (1.7) in the form:

$$U = \sum_{n=0}^{\infty} (n + \tfrac{1}{2})\ \phi(n + \tfrac{1}{2})\ P_n(\cos\theta) \tag{2.1}$$

where

$$\phi(n + \tfrac{1}{2}) = \frac{2i}{kbr}\ \zeta_n(kb) \left[\psi_n(kr) - A_n\ \zeta_n(kr)\right] \tag{2.2}$$

We put $n + \frac{1}{2} = \nu$ and consider ν as a complex variable. The function $\phi(\nu)$ is an analytic function of ν with poles only in the first quadrant. As shown in sec. 2, ref. 1, for the condition $ka \gg 1$ the sum (2.1) can be replaced to a good approximation by the integral

$$U = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi\sin\theta}} \int_C e^{i\nu\theta}\ \phi(\nu)\ \sqrt{\nu}\ d\nu \tag{2.3}$$

where the contour C goes from infinity in the second quadrant, includes all poles of $\phi(\nu)$ and extends to infinity in the first quadrant with the complex variable ν.

The principal portion of the contour will be that in which

$$\nu = ka + (k\,a/2)^{1/3}\ \tau \tag{2.4}$$

while $|\tau|$ is bounded (since ka is assumed very large). The quantity

$$m = (k\,a/2)^{1/3} \tag{2.5}$$

represents the "large parameter" of our problem (we will discard terms of the order of $1/m^2$ in comparison with unity).

This quantity will frequently be encountered in further calculations.

Over most of the integration range we can replace the functions ψ_n and ζ_n by their asymptotic expansions in Airy functions, investigated in detail in ref. 1. We shall consider four Airy functions: $u(\tau)$, $v(\tau)$, $w_1(\tau)$ and $w_2(\tau)$. These functions represent solutions of the differential equation

$$w''(\tau) = \tau\, w(\tau) \qquad (2.6)$$

connected by the relations

$$w_1(\tau) = u(\tau) + iv(\tau) \qquad w_2(\tau) = u(\tau) - iv(\tau) \qquad (2.7)$$

For real τ the functions $u(\tau)$ and $v(\tau)$ are real. The function $w_1(\tau)$ is expressed in terms of the Hankel function of 1st kind and of 1/3 order by the equation

$$w_1(\tau) = e^{i\frac{2\pi}{3}} (\pi/3)^{1/2} (-\tau)^{1/2}\, H^{(1)}_{1/3}\left[\frac{2}{3}(-\tau)^{3/2}\right] \qquad (2.8)$$

Sometimes we will write $w(\tau)$ instead of $w_1(\tau)$.

The asymptotic expansions for the functions ζ_n and their derivatives have the form:

$$\zeta_n(ka) = -\, i m^{1/2}\, w_1(\tau) \qquad \zeta_n'(ka) = i m^{-1/2}\, w_1'(\tau) \qquad (2.9)$$

Taking the real parts, we obtain

$$\psi_n(ka) = m^{1/2}\, v(\tau) \qquad \psi_n'(ka) = -m^{-1/2}\, v'(\tau) \qquad (2.10)$$

The quantity $\chi_n(ka)$ may, according to equation (5.21) in ref. 1, be replaced by the expression

$$\chi_n(k_2a) = -\, i \sqrt{1 - \frac{k^2}{k_2^2}} = -\, i\sqrt{\frac{\eta-1}{\eta}} \qquad (2.11)$$

putting also

$$q = im\frac{\sqrt{\eta - 1}}{\eta} \approx \frac{im}{\sqrt{\eta + 1}} \tag{2.12}$$

and substituting in (1.8) we obtain

$$A_n = i\,\frac{v'(\tau) - qv(\tau)}{w_1'(\tau) - qw_1(\tau)} \tag{2.13}$$

In the equation (2.2) for $\phi(v)$ the functions $\psi_n(kr)$, $\zeta_n(kr)$ and $\zeta_n(kb)$ enter. Their asymptotic expansions may be readily obtained from the foregoing. We set

$$y_1 = \frac{k}{m}\,(r-a) = \frac{kh_1}{m} \tag{2.14}$$

$$y_2 = \frac{k}{m}\,(b-a) = \frac{kh_2}{m} \tag{2.15}$$

where h_2 is the source height, h_1 is the height of the point of observation, and y_2 and y_1 are the corresponding 'reduced elevations'. We then have

$$\zeta_n(kb) = -\,im^{1/2}\,w_1(\tau-y_2) \tag{2.16}$$

$$\zeta_n(kr) = -\,im^{1/2}\,w_1(\tau-y_1) \tag{2.17}$$

$$\psi_n(kr) = m^{1/2}\,v(\tau-y_1) \tag{2.18}$$

and the function $\phi(v)$ is written in the form

$$\phi(v) = \frac{1}{am^2}\,F(\tau,y_1,y_2,q) \tag{2.19}$$

where

$$F = w_1(\tau-y_2)\left\{v(\tau-y_1) - \frac{v'(\tau) - qv(\tau)}{w_1'(\tau) - qw_1(\tau)}\,w_1(\tau-y_1)\right\} \tag{2.20}$$

Expressing v in terms of w_1 and w_2, we can also write

$$F = \frac{1}{2} w_1(\tau - y_2)\left\{w_2(\tau - y_1) - \frac{w_2'(\tau) - qw_2(\tau)}{w_1'(\tau) - qw_1(\tau)} w_1(\tau - y_1)\right\} \tag{2.21}$$

We must now substitute the expression for $\phi(v)$ into (2.3). Introducing the horizontal distance $s = a\theta$, measured along the circumference of the earth, and the 'reduced horizontal distance'

$$x = \left(\frac{k}{2a^2}\right)^{1/3} s = \frac{ms}{a} \tag{2.22}$$

and replacing $v^{1/2}$ in (2.3) by the constant $(ka)^{1/2}$, we obtain

$$U = \frac{e^{iks}}{\sqrt{sa \sin (s/a)}} V (x, y_1, y_2, q) \tag{2.23}$$

where

$$V = e^{-\frac{i\pi}{4}} \left(\frac{x}{\pi}\right)^{1/2} \int e^{ix\tau} F(\tau, y_1, y_2, q)\, d\tau \tag{2.24}$$

This equation is valid for $y_1 < y_2$; if on the other hand $y_1 > y_2$, it is necessary to interchange y_1 and y_2 in the equation for F. The function V may be called the attenuation factor.

3. The Attenuation Factor

Turning now to a study of the attenuation factor, we examine first several limiting cases. We put $y_1 = 0$, corresponding to the case when one of the points (source of point of observation) is located on the earth's surface. We then obtain

$$F(\tau, 0, y_2, q) = \frac{w_1(\tau - y_2)}{w_1'(\tau) - qw_1(\tau)} \tag{3.1}$$

and equation (2.24) for the attenuation factor is reduced to equation (6.02) in ref. 1.

We now consider a second case. We take x and y_2 very large, but the difference

$$x - \sqrt{y_2} = \xi \tag{3.2}$$

as finite. Replacing in (2.21) the function $w_1(\tau - y_2)$ by its asymptotic expansion (equation 8.04 in ref. 1), we have

$$V(x,y_1,y_2,q) = \left(\frac{x^2}{y^2}\right)^{1/4} e^{i\frac{2}{3}y_2^{3/2}} \frac{1}{2\sqrt{\pi}} \int e^{i\xi\tau} \left\{ w_2(\tau - y_2) - \frac{w_2'(\tau) - q w_2(\tau)}{w_1'(\tau) - q w_1(\tau)} w_1(\tau - y_1) \right\} d\tau \tag{3.3}$$

The integral coincides with the expression obtained in our work "The Field of a Plane Wave Near the Surface of a Conducting Body"[3] (equation 4.39). This agreement is entirely understandable. Indeed, for large x and y_2, the source is remote from the point of observation and from the earth's surface, so that a wave, proceeding from the source, may be regarded as plane.

In the general case the integral (2.24) for V may be evaluated as a sum of differences. The function F, defined by (2.20) and (2.21), may be written in the form

$$F = w_1(\tau - y_2) \frac{f(y_1,\tau)}{w_1'(\tau) - q w_1(\tau)} \tag{3.4}$$

where

$$f(y_1,\tau) = \left[w_1'(\tau) - q w_1(\tau)\right] v(\tau - y_1) - \left[v'(\tau) - q v(\tau)\right] w_1(\tau - y_1) \tag{3.5}$$

We note that for $y_1 = 0$ the function f and its derivative take on the values

$$f = 1 \qquad \frac{\partial f}{\partial y_1} = -q \tag{3.6}$$

Hence it is not difficult to conclude that if τ is a root of the equation

$$w_1'(\tau) - q w_1(\tau) = 0 \qquad \tau = \tau_1, \tau_2 \ldots \tag{3.7}$$

then the value of the function f coincides with the value of the expression

$$f(y_1, \tau_s) = f_s(y_1) = \frac{w_1(\tau_s - y_1)}{w_1(\tau_s)} \tag{3.8}$$

which may be called the "height factor".

Evaluating the integral (2.24) as the sum of differences at the points $\tau = \tau_s$, we obtain the following expression for the attenuation factor V: $V(x, y_1, y_2, q) =$

$$e^{\frac{i\pi}{4}} \, 2\sqrt{\pi x} \sum_{s=1}^{\infty} \frac{e^{ix\tau_s}}{\tau_s - q^2} \, \frac{w_1(\tau_s - y_1)}{w_1(\tau_s)} \, \frac{w_1(\tau_s - y_2)}{w_1(\tau_s)} \tag{3.9}$$

which differs from our previous expression for V, corresponding to the case $y_1 = 0$ (see eq'n. 7.06 in ref. 1) only in that now two height factors enter instead of one. The elevations y_1 and y_2 enter symmetrically in (3.9). Using (3.7) we can write (3.9) in the form:

$$V = -e^{\frac{i\pi}{4}} \, 2\sqrt{\pi x} \sum_{s=1}^{\infty} \frac{e^{ix\tau_s}}{1 - \tau_s/q^2} \, \frac{w_1(\tau_s - y_1)}{w_1'(\tau_s)} \, \frac{w_1(\tau_s - y_2)}{w_1'(\tau_s)} \tag{3.10}$$

This form is convenient for q large. In particular, for $q = \infty$ we have

$$V = -e^{\frac{i\pi}{4}} \, 2\sqrt{\pi x} \sum_{s=1}^{\infty} e^{ix\tau_s^o} \, \frac{w_1(\tau_s^o - y_1)}{w_1'(\tau_s^o)} \, \frac{w_1(\tau_s^o - y_2)}{w_1'(\tau_s^o)} \tag{3.11}$$

where the quantities τ_s^o are roots of the equation

$$w_1(\tau_s^o) = 0 \tag{3.12}$$

The series thus obtained are convenient for calculations in the region of shadow. In the illuminated region they converge very slowly, but there one may use a reflection formula, which will be developed in the next section. In the penumbra region one must resort to quadrature for the calculation of V.

4. Reflection Formula

We now consider the field in the illuminated region. We may expect that in this case a reflection formula will be obtained which applies to the reflection of spherical waves from a spherical surface. In the integral (2.24) we may take the expression (2.21) for F. This expression contains two terms. The integrals from each of these terms separately may not converge (only their difference converging) but, applying the method of stationary phase, we can confine ourselves to the consideration of that portion of the integration lying near an extremum of the phase, and may then examine each integral separately.

We put

$$V^o = \frac{1}{2} e^{\frac{i\pi}{4}} \sqrt{\frac{x}{\pi}} \int e^{ix\tau} \, w_1(\tau - y_2) \, w_2(\tau - y_1) \, d\tau \tag{4.1}$$

$$V^* = \frac{1}{2} e^{\frac{i\pi}{4}} \sqrt{\frac{x}{\pi}} \int e^{ix\tau} \, \frac{w_2'(\tau) - qw_2(\tau)}{w_1'(\tau) - qw_1(\tau)} \, w_1(\tau - y_1) \, w_1(\tau - y_2) \, d\tau \tag{4.2}$$

Then the attenuation factor V will be equal to the difference

$$V = V^o - V^* \tag{4.3}$$

Assuming that the most of the range of integration lies in the region of large negative τ and crosses the negative real axis of τ from left above to right below, we can replace w_1 and w_2 by their asymptotic expansions, appropriate to this region. According to equation (8.03) in ref. 1, we may put

$$w_1(\tau-y) = e^{i\frac{\pi}{4}} (y-\tau)^{-1/4} e^{i\frac{2}{3}(y-\tau)^{3/2}} \tag{4.4}$$

$$w_2(\tau-y) = e^{i\frac{\pi}{4}} (y-\tau)^{-1/4} e^{-i\frac{2}{3}(y-\tau)^{3/2}} \tag{4.5}$$

Substituting (4.4) and (4.5) in (4.1) we obtain

$$V^o = \frac{1}{2} e^{i\frac{\pi}{4}} \sqrt{\frac{x}{\pi}} \int e^{i\omega(\tau)} \frac{d\tau}{\left[(y_1-\tau)(y_2-\tau)\right]^{1/4}} \tag{4.6}$$

where

$$\omega(\tau) = x\tau + \frac{2}{3}(y_2-\tau)^{3/2} - \frac{2}{3}(y_1-\tau)^{3/2} \tag{4.7}$$

Evaluating τ from the condition $\omega'(\tau) = 0$, we have

$$\sqrt{y_1-\tau} = \frac{y_2-y_1-x^2}{2x} \; ; \quad \sqrt{y_2-\tau} = \frac{y_2-y_1+x^2}{2x} \tag{4.8}$$

We note that for (4.4) and (4.5) to apply, both quantities (4.8) must be much greater than unity. The value $\omega(\tau)$ at a given τ we denote by ω. This quantity is equal to

$$\omega = \frac{(y_1-y_2)^2}{4x} + \frac{1}{2} x (y_1+y_2) - \frac{1}{12} x^3 \tag{4.9}$$

Application of the method of stationary phase to the integral (4.6) gives

$$V^o = e^{i\omega} \tag{4.10}$$

The quantity ω has a simple geometrical meaning, namely

$$\omega = k(R-s) \tag{4.11}$$

where R is the distance between source and observation point, considered as a straight line, and s is the corresponding horizontal distance, measured along the earth's circumference. From this it is clear that the quantity V^o corresponds to an incident wave.

We now examine the integral V^*. Substituting in (4.2) the asymptotic expansions (4.4) and (4.5), we obtain

$$V^* = \frac{1}{2} e^{\frac{i\pi}{4}} \sqrt{\frac{x}{\pi}} \int e^{i\phi(\tau)} \frac{q-i\sqrt{-\tau}}{q+i\sqrt{-\tau}} \frac{d\tau}{\left[(y_1-\tau)(y_2-\tau)\right]^{1/4}} \tag{4.12}$$

where

$$\phi(\tau) = x\tau + \frac{2}{3}(y_1 - \tau)^{1/2} + \frac{2}{3}(y_2-\tau)^{1/2} - \frac{4}{3}(-\tau)^{1/2} \tag{4.13}$$

We denote by $\tau = -p^2$ the root of the equation $\phi'(\tau) = 0$; where $p > 0$. The quantity p is the root of the equation

$$\sqrt{y_1+p^2} + \sqrt{y_2+p^2} = 2p + x \tag{4.14}$$

which is reduced to a cubic equation. We denote by ϕ the value of the phase $\phi(\tau)$ at $\tau = -p^2$. Using (4.14) we can eliminate all the radicals except p and write ϕ in the form:

$$\phi = -3p^2 x + 2p(y_1+y_2-x^2) + x(y_1+y_2) - \frac{1}{3} x^3 \tag{4.15}$$

Evaluation of the integral V^* by the method of stationary phase gives

$$V^* = \frac{q-ip}{q+ip} \sqrt{A}\; e^{i\phi} \tag{4.16}$$

where

$$A = \frac{px}{3px + x^2 - y_1 - y_2} \tag{4.17}$$

The equation obtained has a simple geometrical interpretation. The quantity p is expressed as

$$p = m \cos \gamma = \left(\frac{ka}{2}\right)^{1/3} \cos \gamma \qquad (4.18)$$

where γ is the angle of incidence of the beam (Fig. 1).

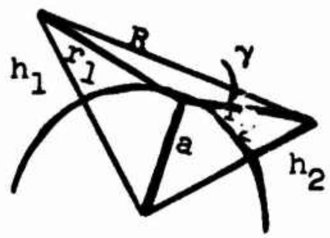

Fig. 1

The factor $(q - ip)/(q + ip)$ is the Fresnel coefficient with reversed sign. A^2 is the product of R/r_1 times the correction for spreading of the bundle of rays after reflection. The phase ϕ is approximately given by

$$\phi = k(r_1 + r_2 - s) \qquad (4.19)$$

where r_2 is the path traversed by the beam after reflection. The expression for the integral V thus obtained by the method of stationary phase

$$V = e^{i\omega} - \frac{q-ip}{q+ip} \sqrt{A}\; e^{i\phi} \qquad (4.20)$$

agrees exactly with the reflection formula. It must be emphasized that this expression (and consequently also the reflection formula) is valid only under the condition that p be sufficiently large compared with unity (it is sufficient to require that p be greater than 2, or better, $p > 3$).

If x, y_1, and y_2 are given, then p is determined from (4.14). This equation can be most simply solved in the following way. We introduce a new unknown, z, putting

$$\sqrt{y_1 + p^2} - p = \frac{1}{2}(x + z) \tag{4.21}$$

$$\sqrt{y_2 + p^2} - p = \frac{1}{2}(x - z) \tag{4.22}$$

Solving (4.21) for p, we obtain

$$p = \frac{y_1}{x + z} - \frac{1}{4}(x + z) \tag{4.23}$$

while (4.22) gives

$$p = \frac{y_2}{x - z} - \frac{1}{4}(x - z) \tag{4.24}$$

Adding the two expressions for p, we obtain a cubic equation

$$z^3 - z(x^2 + 2y_1 + 2y_2) + 2x(y_1 - y_2) = 0 \tag{4.25}$$

which is not difficult to solve. We set

$$\rho^2 = \frac{1}{3}(x^2 + 2y_1 + 2y_2); \quad (\rho > 0) \tag{4.26}$$

$$\sin\alpha = \frac{x(y_1 - y_2)}{\rho^3} \quad \left(-\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right) \tag{4.27}$$

Then the root of (4.25) in which we are interested is

$$z = 2\rho \sin(\alpha/3) \tag{4.28}$$

Using the cubic equation, one may write an expression for p in terms of z in the form

$$2px = y_1 + y_2 - \frac{1}{2}(x^2 + z^2) \tag{4.29}$$

The expression for V can be simplified if we introduce the quantity

$$p_1 = \frac{x^2 - z^2}{2x} = \frac{2px + x^2 - y_1 - y_2}{x} \tag{4.30}$$

We then have

$$V = e^{i\omega}\left(1 - \frac{q - ip}{q + ip}\sqrt{\frac{p}{p + p_1}}\right) e^{2ip_1p^2} \tag{4.31}$$

We note that x,z, and p_1 can be written approximately as

$$x = \left(\frac{k}{2a^2}\right)^{1/3}(r_1 + r_2)\ , \quad z = \left(\frac{k}{2a^2}\right)^{1/3}(r_1 - r_2)$$

$$p_1 = \left(\frac{k}{2a^2}\right)^{1/3} \frac{2r_1r_2}{(r_1 + r_2)} \tag{4.32}$$

If $y_1 = 0$,

$$z = -x, \quad p = (y_2 - x^2)/2x; \quad p_1 = 0 \tag{4.33}$$

If $y_1 = y_2 = y$, we have

$$z = 0, \quad p = y/x - \frac{x}{4} \quad p_1 = \frac{x}{2} \tag{4.34}$$

We now set $x - \sqrt{y_2} = \xi$ as in (3.2) and increase x and $\sqrt{y_2}$, holding ξ finite. This corresponds to the transition to an incident plane wave. Putting for brevity

$$\sqrt{\xi^2 + 3y_1} = \sigma \tag{4.35}$$

we have

$$z = -x + \frac{2}{3}(\sigma + \xi), \quad p = \frac{1}{3}(\sigma - 2\xi), \quad p_1 = \frac{2}{3}(\sigma + \xi) \tag{4.36}$$

and equation (4.31) gives an expression for the integral of (3.3) which coincides with that obtained in § 6, ref. 3, for the case of a plane wave (Note: The factor 2/27 in equation 6.19 of ref. 3 should be changed to 4/27).

5. Horizontal Electrical Dipole. Primary Field.

The field of an electrical dipole may be written in the form:

$$\bar{E} = \text{grad div } \Pi - \Delta\Pi, \quad \bar{H} = - ik \text{ curl } \Pi \tag{5.1}$$

where Π is the Hertz vector, directed along the axis of the dipole and proportional to the quantity

$$\Pi_o = e^{ikR}/R \tag{5.2}$$

where

$$R = \sqrt{b^2 + r^2 - 2br \cos \theta} \tag{5.3}$$

Our spherical coordinates are connected to cartesian coordinates by the relations

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta \tag{5.4}$$

Taking the dipole, located at $x = 0$, $y = 0$, $z = b$, as directed along the x axis, and setting the coefficient of proportionality between Π_x and Π_o as unity, we can write

$$\Pi_x = \Pi_o, \quad \Pi_y = 0, \quad \Pi_z = 0 \tag{5.5}$$

The field components are obtained after substituting the vector (5.5) into equation (5.1). In the following we shall need only the radial components of the primary dipole field or the quantities rE_r and rH_r, proportional to them, which because of the conditions

$$\text{div } \bar{E} = 0, \quad \text{div } \bar{H} = 0 \tag{5.6}$$

are solutions of the scalar wave equation. (5.1) and (5.5) give the following expression for rE_r:

$$rE_r = \cos\phi \frac{\partial}{\partial\theta}\left(\cos\theta \frac{\partial\Pi_o}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\Pi_o}{\partial\theta}\right) -$$

$$\cos\phi\left(\sin\theta \frac{\partial\Pi_o}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\Pi_o}{\partial\theta}\right) \qquad (5.7)$$

Since Π_o is related to r,θ, and b only through R, we have

$$\cos\theta \frac{\partial\Pi_o}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\Pi_o}{\partial\theta} = -\frac{\partial\Pi_o}{\partial b} \qquad (5.8)$$

$$\sin\theta \frac{\partial\Pi_o}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\Pi_o}{\partial\theta} = \frac{1}{b}\frac{\partial\Pi_o}{\partial b} \qquad (5.9)$$

Therefore we can write instead of (5.7)

$$rE_r^o = -\cos\phi \frac{\partial}{\partial\theta}\left(\frac{\partial\Pi_o}{\partial b} + \frac{\Pi_o}{b}\right) \qquad (5.10)$$

The quantity rH_r is immediately obtained from (5.1) in the form

$$rH_r^o = ik \sin\phi \frac{\partial\Pi_o}{\partial\theta} \qquad (5.11)$$

In the last two equations we used the superscript o to emphasize the fact that these equations refer to the primary field.

On the other hand, the complete field can be expressed by means of two auxiliary functions u and v according to the equations

$$E_r = \frac{1}{r}\Delta^* u$$

$$E_\theta = -\frac{1}{r}\frac{\partial^2(ru)}{\partial r\,\partial\theta} + \frac{i\omega}{c\sin\theta}\frac{\partial v}{\partial\phi} \qquad (5.12)$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial^2(ru)}{\partial r\,\partial\phi} - i\frac{\omega}{c}\frac{\partial v}{\partial\theta},$$

$$H_r = -\frac{1}{r}\Delta^* v$$

$$H_\theta = \frac{ick^2}{\omega \sin\theta}\frac{\partial u}{\partial \phi} + \frac{1}{r}\frac{\partial^2(rv)}{\partial r\,\partial\theta} \tag{5.13}$$

$$H_\phi = -i\frac{ik^2}{\omega}\frac{\partial u}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial^2(rv)}{\partial r\,\partial\phi}$$

where Δ^* is the Laplace operator on the sphere

$$\Delta^* u = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial y^2} \tag{5.14}$$

The functions u and v may be considered as electrical and magnetic Hertz functions; sometimes they are called Debye potentials. Both of these functions satisfy the scalar wave equation.

The equations (5.12) give both the field in air and in the earth. For air one must put for the earth $k = k_2 = (\omega/c)\sqrt{\eta}$ where ω is the frequency; in the following we will take k to mean the value of this quantity for air.

The boundary conditions for the electrical and magnetic Hertz function arise from the continuity of the tangential components of the field at the earth's surface. Denoting by the superscript 2 those quantities which refer to the field in the earth, we can write the boundary conditions in the form:

$$k^2 u = k_2^2 u^{(2)} \;;\quad \frac{\partial(ru)}{\partial r} = \frac{\partial\left[ru^{(1)}\right]}{\partial r} \;;\quad r = a \tag{5.15}$$

$$v = v^{(1)} \;;\quad \frac{\partial(rv)}{\partial r} = \frac{\partial\left[rv^{(1)}\right]}{\partial r} \;;\quad r = a \tag{5.16}$$

We must find the form of the functions $u = u^o$ and $v = v^o$, corresponding to the primary field in air. Adding the expressions for rE_r^o from (5.10) and (5.12), we obtain

$$\Delta^* u = -\cos\phi\frac{\partial}{\partial\theta}\left(\frac{\partial\Pi_o}{\partial b} + \frac{\Pi_o}{b}\right) \tag{5.17}$$

and analogously

$$\Delta^* \, v^o = - ik \sin\phi \frac{\partial \Pi_o}{\partial \theta} \tag{5.18}$$

Out of these relations it is easy to determine u^o and v^o if we write Π_o in series form

$$\Pi_o = \frac{e^{ikR}}{R} = \frac{i}{kbr} \sum_{n=0}^{\infty} (2n+1) \, \zeta_n (kb) \, \psi_n (kr) \, P_n (\cos\theta) \tag{5.19}$$

valid for $r < b$. The results are more conveniently written as

$$u^o = - \cos\phi \frac{\partial P}{\partial \theta} \qquad v^o = - \sin\phi \frac{\partial Q^o}{\partial \theta} \tag{5.20}$$

where P^o and Q^o are new auxiliary functions which also satisfy the scalar wave equation, but do not depend on the angle. We have

$$P^o = - \frac{i}{br} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \, \zeta_n' (kb) \, \psi_n (kr) \, P_n (\cos\theta) \tag{5.21}$$

$$Q^o = \frac{1}{br} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \, \zeta_n (kb) \, \psi_n (kr) \, P_n (\cos\theta) \tag{5.22}$$

6. Series for the Total Field

We represent the functions u and v, in terms of which, according to (5.12) and (5.13), the total field is expressed, in the form:

$$u = - \cos\phi \frac{\partial P}{\partial \theta} \; ; \quad v = - \sin\phi \frac{\partial Q}{\partial \theta} \tag{6.1}$$

where P and Q satisfy the scalar wave equation and the following boundary conditions, arising from (5.15) and (5.16):

$$k^2 P = k_2^2\, P^{(2)} \; ; \quad \frac{\partial (rP)}{\partial r} = \frac{\partial \left[r P^{(2)} \right]}{\partial r} \; ; \quad r = a \tag{6.2}$$

$$Q = Q^{(2)} \; ; \quad \frac{\partial (rQ)}{\partial r} = \frac{\partial \left[r Q^{(2)} \right]}{\partial r} \; ; \quad r = a \tag{6.3}$$

and do not depend on the angle ϕ. Keeping in mind the form of the primary excitation (5.21), we can write series for P in air and in the earth as follows:

$$P = - \frac{i}{br} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \zeta_n' \,(kb) \left[\psi_n(kr) - A_n\, \zeta_n(kr) \right] P_n(\cos \theta) \tag{6.4}$$

$$(a < r < b)$$

$$P^{(1)} = - \frac{i}{br} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \zeta_n' \,(kb)\, A_n'\, \psi_n \,(k_2 r)\, P_n(\cos \theta) \tag{6.5}$$

$$0 < r < a$$

The boundary conditions (6.2) give for the coefficients A_n and A_n' the equations

$$k^2 A_n\, \zeta_n\,(ka) + k^2 A_n'\, \psi_n\,(k_2 a) = k^2\, \psi_n\,(ka)$$

$$k\;\; A_n\, \zeta_n'\,(ka) + k_2\, A_n'\, \psi_n'\,(k_2 a) = k\, \psi_n'\,(ka) \tag{6.6}$$

from which

$$A_n = \frac{k_2\, \psi_n'\,(ka)\, \psi_n\,(k_2 a) - k\, \psi_n\,(ka)\, \psi_n'\,(k_2 a)}{k_2\, \zeta_n'\,(ka)\, \psi_n\,(k_2 a) - k\, \zeta_n\,(ka)\, \psi_n'\,(k_2 a)} \tag{6.7}$$

$$A_n' = \frac{ik^2/k_2}{k_2\, \zeta_n'\,(ka)\, \psi_n\,(k_2 a) - k\, \zeta_n\,(ka)\, \psi_n'\,(k_2 a)} \tag{6.8}$$

We note that the coefficient A_n here is exactly the same as in the series (1.7) for the Hertz function U of a vertical dipole. Comparison of the series (1.7) and (6.4) for U and for P shows that these functions are connected by the relation

$$\Delta^* P = \frac{\partial U}{\partial b} + \frac{U}{b} \tag{6.9}$$

This connection between P and U permits us in the following to use the results at hand for the summation of series and to express P in terms of the attenuation factor V which we have already studied.

In analogous manner we can obtain series for the function Q in air and in the earth. Remembering (5.22), we can write

$$Q = \frac{1}{br} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \zeta_n(kb) \left[\psi_n(kr) - B_n \zeta_n(kr) \right] P_n(\cos \theta)$$

$$(a < r < b) \tag{6.10}$$

$$Q^{(2)} = \frac{1}{br} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \zeta_n(kb) B_n' \psi_n(k_2 r) P_n(\cos \theta)$$

$$(0 < r < a) \tag{6.11}$$

The boundary conditions (6.3) give

$$\begin{aligned} B_n \zeta_n(ka) + B_n \psi_n(k_2 a) &= \psi_n(ka) \\ kB_n \zeta_n'(ka) + k_2 B_n' \psi_n'(k_2 a) &= k\psi_n'(ka) \end{aligned} \tag{6.12}$$

from which

$$B_n = \frac{k\psi_n'(ka)\ \psi_n(k_2a) - k_2\psi_n(ka)\psi_n'(k_2a)}{k\zeta_n'(ka)\ \psi_n(k_2a) - k_2\zeta_n(ka)\psi_n'(k_2a)} \tag{6.13}$$

$$B_n' = ik/\ k\zeta_n'(ka)\ \psi_n(k_2a) - k_2\ \zeta_n(ka)\ \psi_n'(k_2a) \tag{6.14}$$

Thus we have determined series for the function Q. We note that Q is connected with the Hertz function for a vertical magnetic dipole (horizontal space antenna). The field from such a dipole can be represented by equations (5.12) and (5.13), where we must put

$$u = 0\ , \qquad v = \frac{M}{b}\,W \tag{6.15}$$

where the factor M is the magnetic moment, and the function W is of the same nature as Π_o and satisfies the same boundary conditions (5.16) as v.

For the function W in air we obtain an expansion of the form

$$W = \frac{1}{kbr}\sum_{n=0}^{\infty}(2n+1)\ \zeta_n(kb)\left[\psi_n(kr) - B_n\ \zeta_n(kr)\right]P_n(\cos\theta) \tag{6.16}$$

where B_n is given by (6.13). Comparison of the series (6.10) and (6.16) for Q and for W yields the relatior

$$\Delta^*Q = ik\ W \tag{6.17}$$

The same relation (with the same value of $k = \omega/c$) is obtained for the value of these functions in the earth.

7. Approximate Expressions for the Field

Series for our functions P and Q are constructed analogously to the series for U which was summed approximately in the preceding sections. In addition, P is related to U through (6.9). Therefore, it is not necessary to repeat the arguments which led us to the summation of the series for U, and we can use the results already obtained. For the determination of P we use the relation

$$\Delta^* P = \frac{\partial U}{\partial b} + \frac{U}{b} \tag{7.1}$$

and the value (2.22) for U:

$$U = \frac{e^{iks}}{\sqrt{sa \sin(s/a)}} \quad V(x,y_1,y_2,q) \tag{7.2}$$

It can be readily seen that in those approximations in which (7.2) is valid, the application of the operator Δ^* to functions of the type U or P is equivalent to multiplication by $-k^2a^2$. On the other hand, on the right side of (7.1) we can neglect the term U/b in comparison with the derivative $\partial U/\partial b$ and express this derivative according to (2.15) as the derivative with respect to y_2. (7.1) then gives

$$-k^2a^2 \; P = \frac{k}{m} \frac{\partial U}{\partial y_2} \tag{7.3}$$

from which

$$P = - \frac{1}{ka^2m} \; \frac{\partial U}{\partial y_2} \tag{7.4}$$

Analogously, one can express Q in terms of W on the basis of (6.17). We obtain

$$Q = - \; (1/ka^2) \; W \tag{7.5}$$

We have already derived an approximate expression (7.2) for U. An analogous expression can be derived for W. The series (6.16) for W differs from the series (1.7) for U only in that the coefficient A_n, determined by (1.8), is replaced in it by the coefficient B_n, determined by (6.13). With the same degree of accuracy to which (2.12) is valid, we can write

$$B_n = i\,\frac{v'(\tau) - q_1 v(\tau)}{w_1'(\tau) - q_1 w_1(\tau)} \tag{7.6}$$

where

$$q' = q\eta = im(\eta - 1)^{1/2} \tag{7.7}$$

Therefore W differs from U only by the substitution of q for q_1, and we have

$$W = \frac{e^{iks}}{\sqrt{sa\,\sin(s/a)}}\quad V(x, y_1, y_2, q_1) \tag{7.8}$$

In practice one can put $q_1 = \infty$ in all cases. Then the series for V acquires the form (3.11).

We must now substitute the expressions obtained into the equations for the field. To do this, we find first the electrical and magnetic Hertz functions u and v. On the basis of (6.1) we have

$$u = \frac{i}{am}\,\frac{\partial U}{\partial y_2}\quad \cos\phi \tag{7.9}$$

$$v = -\frac{1}{a}\,W \sin\phi \tag{7.10}$$

We substitute these expressions into (5.12) and (5.13), retaining only the important terms and neglecting quantities of order $1/m^2$ compared with unity. We then obtain the following simple expressions:

$$E_r = -k^2 au = -\frac{ik^2}{m}\frac{\partial U}{\partial y_2}\cos\phi$$

$$E_\theta = 0 \qquad (7.11)$$

$$E_\phi = k^2 av = -k^2 W \sin\phi$$

$$H_r = k^2 av = -k^2 W \sin\phi$$

$$H_\theta = \frac{ik^2 a}{m}\frac{\partial v}{\partial y_1} = -\frac{ik^2}{m}\frac{\partial W}{\partial y_1}\sin\phi \qquad (7.12)$$

$$H_\phi = k^2 au = \frac{ik^2}{m}\frac{\partial U}{\partial y_1}\cos\phi$$

These expressions give the field in 'reduced' units (the moment of the electrical dipole is taken as unity). To obtain the field in conventional units, the expressions must be multiplied by the magnitude of the electrical moment.

We now compare the relations (7.12) with those for a vertical magnetic dipole with unit moment. In accordance with (6.15) we put $u = 0$; $bv = W$, obtaining

$$E_r = 0; \quad E_\theta = 0; \quad E_\phi = k^2 W \qquad (7.13)$$

$$H_r = k^2 W; \quad H_\theta = \frac{ik^2}{m}\frac{\partial W}{\partial y_1}; \quad H_\phi = 0 \qquad (7.14)$$

Thus in the plane perpendicular to the electric dipole, its field either coincides with the field of a vertical magnetic dipole ($\phi = 3\pi/2$), or differs from it in sign ($\phi = \pi/2$).

In conclusion we may make a remark about the character of the field at different distances from the source. At finite values of reduced horizontal distance x, the functions U and W are of the same order. Since (7.11) and (7.12) contain the large parameter m, then at such distances the different field components will not be of the same order; the electrical field will be almost horizontal and the magnetic field almost vertical (the ratio of "small" components to "large" will be of the order of 1/m). However, in the region of deep shadow W will decrease faster than U. Indeed, the decrease of these functions is characterized by the factors

$$e^{ix\tau_1^o} \text{ (for W)} \quad \text{and} \quad e^{ix\tau_1'} \text{ (for U)}$$

where τ_1^o and τ_1' are the roots of the following equation which have the smallest moduli:

$$W_1'(\tau_1^o) - q_1 W_1(\tau_1^o) = 0, \quad W_1'(\tau_1') - qW_1(\tau_1') = 0$$

For soil with good conductivity we may set $q = 0$; $q_1 = \infty$ whence

$$\tau_1^o = 2.338\ e^{i\pi/3} \qquad \tau_1' = 1.019\ e^{i\pi/3}$$

so that the imaginary part of τ_1^o will be larger than the imaginary part of τ_1'. The same relation holds in the general case, because $|q_1|$ is always much larger than $|q|$. Therefore the attenuation of W will occur more rapidly than the attenuation of U, and for sufficiently large x, the terms comprising U can, in spite of the small factor 1/m, predominate over the terms comprising W. This denotes for the electrical field a constant transition from horizontal to vertical polarization (with the reverse transition for the magnetic field).

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IX. FRESNEL DIFFRACTION FROM CONVEX BODIES

V. A. Fock

The method of approximation based on Huyghens' principle for computation of the diffraction permits us, as is well known, to determine the field of a wave which is diffracted by a thin opaque screen. This field is expressed by Fresnel's integrals.

However, in the case where the diffracting body has a finite curvature (radius of curvature large with respect to wavelength) the problem of an approximate determination of the field in the region of the geometrical shadow boundary at large distance from the body has been unanswered up to now; in particular it could not be clarified whether in this case the same expressions for the field (the Fresnel integrals), with which one starts in analogy to the case of an infinitely thin screen, are applicable. In the following paper we show in the example of diffraction from a sphere, that also for a body with finite curvature the main term in the expression for the field behind this body is given by the Fresnel integrals. This term does not depend on the material of the body (in the same way as in the usual Fresnel Diffraction). To the main term there is added here an additional term, which constitutes a sort of background, above which the Fresnel zones lie. This additional term (and with it the background) depends, on the contrary, on the electrical properties of the body from which the wave is diffracted.

1. Formulas for The Attenuation Factor

We start out from our formulas for diffraction, which were developed in our paper [1] . We have to summarize here the main results of this paper.

The field of a light point source (dipole), located at some distance from the surface of the sphere, is given by two functions U and W, which represent solutions of the wave equation.

$$\Delta U + k^2 U = 0 \qquad (1.01)$$

and becomes singular at the source-point in such a manner that

$$U = \frac{e^{ikR}}{R} + U^{o} , \qquad (1.02)$$

Where R is the distance from the source and U^{o} remains finite for $kR \to 0$.

The equations defining U, W differ in the form of the boundary conditions, which we are not going to discuss here.

Let r, θ, ϕ be polar coordinates with origin at the center of the sphere, and axis in direction of the dipole. The quantity $s = a\theta$, where a is the radius of the sphere, gives the distance from the source to the observation point, computed on a large circle. The height of the source above the sphere is denoted by h_1, the height of the observation point - by h_2. Further we introduce the parameter

$$m = \sqrt[3]{\frac{ka}{2}} \qquad (1.03)$$

which we assume large, and let

$$x = \sqrt[3]{\frac{k}{2a^2}}\, s = m \frac{s}{a} = m\theta \; ; \qquad (1.04)$$

$$y_1 = \frac{kh_1}{m} \; ; \quad y_2 = \frac{kh_2}{m} \qquad (1.05)$$

The complex dielectric constant of the material of the sphere we designate by η, and assume $|\eta| \gg 1$. Finally we let

$$q = \frac{im}{\sqrt{\eta + 1}} ; \quad q_1 = im \sqrt{\eta - 1} . \tag{1.06}$$

In our paper we showed, that in the vicinity of the surface of the sphere (that is, at distances small compared to its radius), the functions U and W are to be expressed by an attenuation factor, in accordance with:

$$U = \frac{e^{iks}}{\sqrt{sa \sin \frac{s}{a}}} \cdot V(x, y_1, y_2, q) \tag{1.07}$$

$$W = \frac{e^{iks}}{\sqrt{sa \sin \frac{s}{a}}} \cdot V(x, y_1, y_2, q_1) . \tag{1.08}$$

The attenuation factor V may be represented for $y_1 < y_2$, by the contour integral:

$$V(x, y_1, y_2, q) = e^{-i\frac{\pi}{4}} \sqrt{\frac{x}{\pi}} \int_C c^{ixt} F(t, y_1, y_2, q)\, dt , \tag{1.09}$$

where the function F may be written in the form:

$$F = w_1(t - y_2) \left\{ v(t - y_1) - \frac{v'(t) - qv(t)}{w_1'(t) - qw_1(t)} w_1 (t - y_1) \right\}, \tag{1.10}$$

or in the form:

$$F = \frac{i}{2} w_1(t - y_2) \left\{ w_2(t - y_1) - \frac{w_2'(t) - qw_2(t)}{w_1'(t) - qw_1(t)} w_1 (t - y_1) \right\} \tag{1.11}$$

Here $w_1(t)$ and $w_2(t)$ are the complex Airy-functions, which represent solutions of the differential equation

$$w''(t) = tw(t) \tag{1.12}$$

and which tend for large negative t asymptotically to the following expressions:

$$\left.\begin{aligned} w_1(t) &\sim e^{i\frac{\pi}{4}}(-t)^{-\frac{1}{4}}e^{i\frac{2}{3}(-t)^{3/2}}, \\ w_2(t) &\sim e^{-i\frac{\pi}{4}}(-t)^{-\frac{1}{4}}e^{-i\frac{2}{3}(-t)^{3/2}}. \end{aligned}\right\} \tag{1.13}$$

In the formula (1.10) there appears also one of the functions u(t), v(t), which are defined by the equations

$$w_1(t) = u(t) + iv(t) \; ; \quad w_2(t) = u(t) - iv(t) \tag{1.14}$$

For t real both functions u(t), v(t) are real. For all values of t we have:

$$w_1\left(te^{i\frac{2}{3}\pi}\right) = e^{i\frac{\pi}{3}}w_2(t) \; ; \quad w_1\left(te^{i\frac{4}{3}\pi}\right) = 2e^{i\frac{\pi}{6}}v(t) \; . \tag{1.15}$$

The contour C of the integral (1.09) encloses in positive direction the first quadrant of the complex variable t (in this first quadrant all the poles of the integrand are located). We can choose for contour C a broken line, which goes from $\infty\, e^{i\pi\frac{2}{3}}$ to 0 and from 0 to ∞.

2. Reformulation of The Attenuation Factor

In our previous pages [1, 2] we investigated the attenuation factor V, first in the illuminated region, where the formula of reflection, corresponding to geometrical optics holds, second in the shadow region, where the amplitudes of the field decay exponentially, and finally in the transition region (region of half-shadow). The region of the shadow-cone was not investigated, and it is the purpose of the present work to derive approximations for this region.

The shadow cone is that cone, tangent to the sphere, whose apex is the source point. The equation of the shadow cone may be written in the form

$$\sqrt{b^2 - a^2} + \sqrt{r^2 - a^2} = \sqrt{r^2 + b^2 - 2rb\cos\theta}\,, \quad (2.01)$$

or, after transition to the variables x, y_1, y_2, and neglect of small quantities:

$$\sqrt{y_1} + \sqrt{y_2} = x\,. \quad (2.02)$$

Thus, we have the task of investigating the attenuation factor V for the case, where the quantities x, y_1, y_2 are very large, but the difference

$$\xi = x - \sqrt{y_1} - \sqrt{y_2} \quad (2.03)$$

remains finite. We note that the shadow region corresponds to positive values of ξ, the illuminated region to negative values of ξ.

Under the integral (1.09) for V we may substitute for F one of the two expressions (1.10) or (1.11), which are identical. We decompose the contour C of the integral (1.09) into two segments: that from $\infty e^{\frac{3}{2\pi i}}$ to 0 we denote C_1, that from 0 to ∞ - C_2. On the first segment we use for F the expression (1.11), on the second segment the expression (1.10). Then we may write

$$V = \Phi + \Psi \, , \tag{2.04}$$

where

$$\left.\begin{aligned} \Phi = \sqrt{\frac{x}{2}}\, e^{-i\frac{\pi}{4}} \Bigg\{ \frac{1}{2} \int_{C_1} & e^{ixt}\, w_1(t - y_2)\, w_2(t - y_1)\, dt \, + \\ + \int_{C_2} & e^{ixt}\, w_1(t - y_2)\, v(t - y_1)\, dt \Bigg\} , \end{aligned}\right\} \tag{2.05}$$

$$\begin{aligned} \Psi = -\sqrt{\frac{x}{\pi}}\, e^{-i\frac{\pi}{4}} \Bigg\{ \frac{1}{2} \int_{C_1} & e^{ixt} \frac{w_2'(t) - q w_2(t)}{w_1'(t) - q w_1(t)}\, w_1(t - y_1)\, w_1(t - y_2)\, dt \, + \\ + \int_{C_2} & e^{ixt} \frac{v'(t) - q v(t)}{w_1'(t) - q w_1(t)}\, w_1(t - y_1)\, w_1(t - y_2)\, dt \Bigg\} . \end{aligned} \tag{2.06}$$

The integrals which enter into Φ do not depend on the parameter q, which appears only in Ψ. Consequently, Φ does not depend on the electrical properties of the diffracting body; they influence only the quantity Ψ. We see that Φ corresponds to the Fresnel portion of the diffraction and Ψ to the background, on which the Fresnel Diffraction term is superimposed.

3. Computation of The Integral Φ

In the expression (2.05) for Φ we may replace the integration over C_1 by an integration from $-\infty$ to 0. Using the relation $w_2 = w_1 - 2iv$ we obtain:

$$\Phi = \Phi_1 + \Phi_2 , \tag{3.01}$$

where

$$\Phi_1 = \frac{1}{2}\sqrt{\frac{x}{\pi}}\, e^{i\frac{\pi}{4}} \int_{-\infty}^{0} e^{ixt}\, w_1(t - y_2)\, w_1(t - y_1)\, dt , \tag{3.02}$$

$$\Phi_2 = \sqrt{\frac{x}{\pi}}\, e^{-i\frac{\pi}{4}} \int_{-\infty}^{+\infty} e^{ixt}\, w_1(t - y_2)\, v(t - y_1)\, dt . \tag{3.03}$$

First we compute the integral Φ_2. For this purpose we make use of the following integral-representation for $w_1(t - y_2)$:

$$w_1(t - y_2) = \frac{1}{\sqrt{\pi}} \int_{\Gamma} e^{(t-y_2)\, z - \frac{1}{3} z^3}\, dz , \tag{3.04}$$

Where the curve Γ consists of the segments from $-i\,\infty$ to 0 and from 0 to ∞. We note that on the curve Γ we have: $\mathrm{Re}\, z \geq 0$. Substituting (3.04) and (3.03), we can carry thru the integration over t with help of the formula:

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{(z + ix)t}\, v(t - y_1)\, dt = \exp\left\{y_1(z + ix) + \frac{1}{3}(z + ix)^3\right\} \tag{3.05}$$

which holds for Rez $\geqslant$ 0. Thereby we find

$$\Phi_2 = \sqrt{\frac{x}{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{-\frac{1}{3}x^3 + ixy_1}\, , \int_{\Gamma} e^{ixz^2 - (x^2 + y_2 - y_1)z}\, dz\,. \tag{3.06}$$

The latter integral is easily computed, and we obtain finally

$$\Phi_2 = e^{i\omega(x)}\,, \tag{3.07}$$

with

$$\omega(x) = -\frac{1}{12}x^3 + \frac{1}{2}x\,(y_1 + y_2) + \frac{(y_2 - y_1)^2}{4x}\,. \tag{3.08}$$

As we showed in our paper [1], the quantity ω is the phase of the incident wave, and we have approximately:

$$\omega = k(R - s)\,, \tag{3.09}$$

where R and s designate the same quantities as in Section 1. Thus the integral Φ_2 corresponds to the incident wave.

We now pass to the evaluation of the integral Φ_1. Using the integral-representation (3.04) for both factors $w_1(t - y_2)$ and $w_1(t - y_1)$ we arrive, carrying out the integration over t, at a double contour integral, for which one integration may be carried thru after a change of variables. Thereby we get:

$$\Phi_1 = \frac{\sqrt{x}}{2\pi i} \int_C e^{i\omega(z)} \frac{dz}{\sqrt{z}\,(z - x)} , \qquad (3.10)$$

Where the contour C comes from positive imaginary infinity, intersects the real axis to the right of the point $z = x$, and then proceeds along the ray arc $z = -\frac{\pi}{6}$.

The value of the integral (3.10) at the point $z = x$ is according to (3.07) equal to the quantity Φ_2. Therefore, we get, if we denote by C_1 a contour, which is similar to C, but cuts the real axis to the left of the point $z = x$.

$$\Phi = \Phi_1 + \Phi_2 = \frac{\sqrt{x}}{2\pi i} \int_{C'} e^{i\omega(z)} \frac{dz}{\sqrt{z}\,(z - x)} . \qquad (3.11)$$

By means of this formula we can express the function Φ approximately by means of Fresnel's integrals. To this purpose we make use of the saddle-point method, where we note, however, that the ratio $\frac{1}{z - x}$ is not a slowly-varying function. If we equate the derivative of the phase $\omega(z)$ to zero, we arrive at the equation:

$$z^4 - 2z^2 (y_1 + y_2) + (y_1 - y_2)^2 = 0 , \qquad (3.12)$$

which has roots

$$z = \pm \sqrt{y_1} \pm \sqrt{y_2} . \qquad (3.13)$$

Of these four roots only the largest positive one is interesting:

$$z_0 = \sqrt{y_1} + \sqrt{y_2} , \qquad (3.14)$$

Since it lies closest to the contour C. We call C_o a contour which resembles C and C', but cuts the real axis at a point $z = z_o$. Using the relation (2.03) we set

$$x - z_o = x - \sqrt{y_1} - \sqrt{y_2} = \xi \; . \tag{3.15}$$

If $\xi < 0$, the contour C_o is equivalent to C and the integral over it yields Φ_1. If $\xi > 0$, then the contour C_o is equivalent to C' and the integral over it yields Φ.

Near $z = z_o$ we have

$$\omega(z) = \omega_o - \mu^2(z - z_o)^2 \; , \tag{3.16}$$

with

$$\omega_o = \omega(z_o) + \frac{2}{3} y_1^{3/2} + \frac{2}{3} y_2^{3/2} \; , \tag{3.17}$$

$$\mu^2 = \frac{\sqrt{y_1 y_2}}{\sqrt{y_1} + \sqrt{y_2}} \; . \tag{3.18}$$

for an approximate evaluation of the integral

$$I = \frac{\sqrt{x}}{2\pi i} \int_{C_o} e^{i\omega(z)} \frac{dz}{\sqrt{z}\,(z - x)} \tag{3.19}$$

We replace the quantity $\sqrt{z}$ by the constant value $\sqrt{z_o}$ and the function $\omega(x)$ by the expression (3.16). If we set

$$z = z_o + pe^{-i\frac{\pi}{4}} \; , \tag{3.20}$$

Then we can integrate over p from $-\infty$ to $+\infty$. Thereby we get

$$I = \sqrt{\frac{x}{z_o}}\, e^{i\omega_o} \cdot \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{-\mu^2 p^2} \frac{dp}{p - \xi e^{i\frac{\pi}{4}}} . \quad (3.21)$$

The latter integral is expressible in terms of Fresnel's integrals, where for $\xi > 0$ and $\xi < 0$ it has different analytic forms, namely:

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{-\mu^2 p^2} \frac{dp}{p - \xi e^{i\frac{\pi}{4}}} = \begin{cases} f(\mu\xi) \text{ für } \xi > 0 , & (3.22) \\ - f(-\mu\xi) \text{ für } \xi < 0 , & (3.23) \end{cases}$$

with

$$f(\alpha) = e^{-i\alpha^2 - i\frac{\pi}{4}} \cdot \frac{1}{\sqrt{\pi}} \int_{\alpha}^{\infty} e^{i\alpha^2}\, d\alpha . \quad (3.24)$$

One sees easily, that

$$f(\alpha) + f(-\alpha) = e^{-i\alpha^2} . \quad (3.25)$$

If we introduce the usual Fresnel Integrals

$$C + iS = \sqrt{\frac{2}{\pi}} \int_0^{\alpha} e^{i\alpha^2}\, d\alpha , \quad (3.26)$$

we may write

$$f(\alpha) = \frac{1}{\sqrt{2}}\, e^{-i\alpha^2 - i\frac{\pi}{4}} \left\{ \left(\frac{1}{2} - C\right) + i\left(\frac{1}{2} - S\right) \right\} . \qquad (3.27)$$

The asymptotic expression for $f(\alpha)$, which holds for large values of α, is

$$f(\alpha) = \frac{1}{2\sqrt{\pi}}\, e^{i\frac{\pi}{4}} \left(\frac{1}{\alpha} - \frac{i}{2\alpha^3} + \ldots \right) . \qquad (3.28)$$

If one expresses the integral I by $f(\alpha)$ and remembers that this integral represents, for $\xi > 0$ the function Φ, for $\xi < 0$ — the function $\Phi_1 = \Phi - \Phi_2$ defined by (3.07), one obtains finally:

$$\Phi = \frac{\sqrt{x}}{\sqrt[4]{y_1 y_2}}\, e^{i\omega_o} \,.\, \mu f(\mu\xi) \qquad (\text{for } \xi > 0) , \qquad (3.29)$$

$$\Phi = e^{i\omega(x)} - \frac{\sqrt{x}}{\sqrt[4]{y_1 y_2}}\, e^{i\omega_o} \,.\, \mu f(-\mu\xi) \qquad (\text{for } \xi < 0) . \qquad (3.30)$$

These expressions hold under the condition that $\sqrt{y_1}$ and $\sqrt{y_2}$ are very large (the quantity μ^2 is of the order of the smaller of these numbers). The quantity ξ may be regarded as finite and small, the product $\mu\xi$ may be an arbitrary (large, finite or small) number. If ξ is very small (and it may have arbitrary sign), then both expressions for Φ practically coincide. This follows from the equations of approximation:

$$\frac{\mu^2 x}{\sqrt{y_1 y_2}} = 1 + \frac{\xi}{\sqrt{y_1} + \sqrt{y_2}} \approx 1 , \qquad (3.31)$$

$$\omega(x) \approx \omega_o - \mu^2 \xi^2 \qquad (3.32)$$

in conjunction with the formula (3.25). For $\xi = 0$ the two expressions for Φ coincide exactly.

4. Evaluation of The Integral

We turn now to the result of the approximation formulas for the integral Ψ. We first inquire about the value of the integral for that case, for which we have computed the value of the integral Φ, namely for the case where the quantities $\sqrt{y_1}$, $\sqrt{y_2}$ (and with these also μ^2) are very large, while $\xi = x - \sqrt{y_1} - \sqrt{y_2}$ remains finite. Under these conditions that portion of the contour of integration, on which the variable t is finite, yields the principal part. For finite t, and y_1 and y_2 large, the product of the function w_1 and the exponential appearing under the integral in (2.06) equals:

$$e^{ixt}\, w_1(t - y_1)\, w_1(t - y_2) = \frac{i}{\sqrt[4]{y_1\, y_2}}\; e^{i\omega_0} \cdot e^{i\xi t}$$

$$\left[1 + \frac{it^2}{4\mu^2} + 0\left(\frac{1}{\mu^3}\right)\right], \tag{4.01}$$

Where for the sake of brevity we used the relation (3.17).

If we substitute this expression into the integral Ψ, we obtain:

$$\Psi = -\frac{\sqrt{x}}{\sqrt[4]{y_1\, y_2}}\; e^{i\omega_0} \left\{g(\xi) - \frac{i}{4\mu^2}\, g''(\xi) + 0\left(\frac{1}{\mu^4}\right)\right\}, \tag{4.02}$$

where

$$g(\xi) = \frac{1}{\sqrt{\pi}} e^{i\frac{\pi}{4}} \left\{ \frac{1}{2} \int_{\infty e^{i\frac{2}{3}\pi}}^{0} e^{i\xi t} \frac{w_2'(t) - qw_2(t)}{w_1'(t) - qw_1(t)} dt + \right.$$

$$\left. + \int_0^{\infty} e^{i\xi t} \frac{v'(t) - qv(t)}{w_1'(t) - qw_1(t)} dt \right\} . \qquad (4.03)$$

Making use of the properties of the Airy function (1.15), we see immediately, that with

$$t = t' e^{i\frac{2\pi}{3}} \qquad (4.04)$$

there follows

$$\frac{1}{2} \frac{w_2'(t) - qw_2(t)}{w_1'(t) - qw_1(t)} = \frac{v'(t') - qe^{i\frac{2}{3}\pi} v(t')}{w_2'(t') - qe^{i\frac{2}{3}\pi} w_2(t')} . \qquad (4.05)$$

Substitution of (4.04) converts the first integral in (4.03) into an integral over the positive-real axis. Omitting the prime on t, we get:

$$g(\xi) = e^{-i\frac{\pi}{12}} . \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-\frac{\xi t}{2}(\sqrt{3}+1)} . \frac{v'(t) - qe^{i\frac{2}{3}\pi} v(t)}{w_2'(t) - qe^{i\frac{2}{3}\pi} w_2(t)} dt +$$

$$e^{i\frac{\pi}{4}} \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{i\xi t} . \frac{v'(t) - q(t)}{w_1'(t) - qw_1(t)} dt . \qquad (4.06)$$

The function $v(t)$ in the numerator decreases rapidly with increasing t, while the functions $w_1(t)$ and $w_2(t)$ in the denominator increase as rapidly. Therefore, both integrals converge very rapidly, and may be evaluated by quadratures. The function $g(\xi)$ may be developed in a Taylor-series in ξ; the coefficients of this series may also be evaluated by quadrature. For large, positive ξ the function $g(\xi)$ has asymptotic behavior:

$$g(\xi) = \frac{e^{i\frac{\pi}{4}}}{2\sqrt{\pi}} \cdot \frac{1}{\xi} , \tag{4.07}$$

i.e. of an expression, which no longer depends on q. The remainder is of order $e^{i\xi t_1}$, where t_1 is the first root of the equation

$$w_1'(t) - qw_1(t) = 0 \tag{4.08}$$

For large negative ξ the asymptotic expression for $g(\xi)$ has the form:

$$g(\xi) = \frac{e^{i\frac{\pi}{4}}}{2\sqrt{\pi}} \cdot \frac{1}{\xi} + \frac{\sqrt{-\xi}}{2} \cdot \frac{q + i\frac{\xi}{2}}{q - i\frac{\xi}{2}} e^{-\frac{i}{12}\xi^2} . \tag{4.09}$$

If we substitute this expression in (4.02), we have to remember, that this formula for Ψ applies only when the correction term, which contains μ^2 in the denominator, is small compared to the main term. Necessary for the applicability of like expressions (4.02) and (4.09) is the condition

$$1 \ll \xi^2 \ll \mu \qquad (\xi < 0) . \tag{4.10}$$

5. The Attenuation Factor in The Region of The Shadow-Cone

In the preceding paragraph we found approximate expressions for the integrals Φ and Ψ, whose sum yields the attenuation factor $V(x,y_1,y_2,q)$. Forming this sum, we obtain for $\xi > 0$.

$$V = \frac{\sqrt{x}}{\sqrt[4]{y_1\, y_2}}\, e^{i\omega_0} \left\{\mu f\ (\mu\xi) - q(\xi) + \frac{1}{4\mu^2}\, g''(\xi)\right\} \tag{5.01}$$

and for $\xi \leqslant 0$

$$V = e^{i\omega(x)} - \frac{\sqrt{x}}{\sqrt[4]{y_1\, y_2}}\, e^{i\omega_0} \left\{\mu f(-\mu\xi) + g(\xi) - \frac{1}{4\mu^2}\, g''(\xi)\right\}. \tag{5.02}$$

These expressions are valid under condition, that the parameter μ, which is determined from the equation

$$\mu^2 = \frac{\sqrt{y_1\, y_2}}{\sqrt{y_1} + \sqrt{y_2}} \tag{5.03}$$

is very large, while the quantity

$$\xi = x - \sqrt{y_1} - \sqrt{y_2} \tag{5.04}$$

is small or finite.

Let us return to the geometrical meaning of these quantities. According to the formulas (1.03) to (1.05) we have:

$$\mu^2 = \sqrt[6]{\frac{2k^2}{a}} \cdot \frac{\sqrt{h_1\, h_2}}{\sqrt{h_1} + \sqrt{h_2}}\ , \tag{5.05}$$

$$\xi = \sqrt[3]{\frac{k}{2a^2}}\ (s - \sqrt{2ah_1} - \sqrt{2ah_2})\ . \tag{5.06}$$

These large values of μ correspond to short wavelengths and are relatively large distances from the surface of the body (the latter should still be small compared to its radii of curvature). The quantity ξ is proportional to the distance taken along (more exactly, parallel to) the surface of the body, from the boundary of the geometrical shadow (the shadow cone). For $\xi < 0$ the magnitude $\mu^2\xi^2$ is approximately equal to the phase difference between the incident and reflected waves. The value $\xi = 0$ corresponds to the boundary of the shadow, positive correspond to the shadow, negative ξ to the illuminated region.

Our formulas give the transition between light and shadow at relatively large distances from the surface of the body. Since the functions f and g and their derivatives with respect to their arguments are, for finite values of these arguments, of order 1, the term $\mu f(\mu\xi)$ yields the main term in (5.01) for large values of μ. This term is proportional to the Fresnel integral. It represents a rapidly-varying function of ξ, since the argument of the Fresnel integral is $\mu\xi$, where μ is a large number. Thus the main term in the expression for V yields the Fresnel diffraction. On this diffraction pattern there is superposed an intensity which is represented by the function $g(\xi)$ and is slowly varying in comparison with the main term. This "background" depends on the material of the diffracting body (since $g(\xi)$ depends on q), while the Fresnel term is independent thereof.

The expression derived here for the attenuation factor should reduce, as going further away from the shadow cone in both directions, to the previously derived formulas for the shadow and the illuminated region. We check this: in the shadow zone we have to have an exponential amplitude decay, in the illuminated region — the reflection formula.

Since we neglected in the formula (5.01) and in the asymptotic expression (4.07) for $g(\xi)$, terms which decrease exponentially for large position ξ, we should get zero in our approximation in the shadow zone. Actually we get from the asymptotic expression (3.28) for the Fresnel-function $f(\alpha)$:

$$\mu f(\mu\xi) = \frac{1}{2\sqrt{\pi}} e^{i\frac{\pi}{4}} \cdot \left(\frac{1}{\xi} - \frac{1}{2\mu^2\xi^3}\right) . \tag{5.07}$$

On the other hand, the formula (4.07) yields:

$$g(\xi) - \frac{1}{4\mu^2} g''(\xi) = \frac{1}{2\sqrt{\pi}} e^{i\frac{\pi}{4}} \cdot \left(\frac{1}{\xi} - \frac{1}{2\mu^2\xi^3}\right) , \tag{5.08}$$

That is, the same expression. Thus for large positive ξ the expression (5.01) for V actually tends to zero in our approximation.

Now we consider large negative values ξ. In the formula (5.02) the first term of the asymptotic expression (4.09) for $g(\xi)$ cancels against $\mu f(-\mu\xi)$, the second terms (which contains the exponential function) yields:

$$V = e^{i\omega(x)} - \frac{\sqrt{x}}{\sqrt[4]{y_1 y_2}} e^{i\omega_o} \cdot \frac{\sqrt{-\xi}}{2} \cdot \frac{q + i\frac{\xi}{2}}{q - i\frac{\xi}{2}} e^{-\frac{i}{12}\xi^3} . \tag{5.09}$$

On the other hand, in the illuminated region the reflection formula

$$V = e^{i\omega} \cdot \left(1 - \frac{q - ip}{q + ip} \cdot \sqrt{\frac{p}{p + p_1}} e^{2ip_1p^2}\right) \tag{5.10}$$

holds, as shown in our previous paper [1] [the formula (4.31) of that paper]. There $\omega = \omega(x)$, and the quantity p (which is proportional to the cosine of the angle of incidence) is defined by the equation:

$$\sqrt{y_1 + p^2} + \sqrt{y_2 + p^2} = 2p + x \tag{5.11}$$

and for p_1 we have

$$p_1 = 2p + x - \frac{1}{x}(y_1 + y_2) \ . \tag{5.12}$$

To the approximation in which formula (5.09) holds, we have

$$p = -\frac{\xi}{2} + \frac{\xi^2}{16\,\mu^2} \approx -\frac{\xi}{2} \ , \tag{5.13}$$

$$p_1 = 2\,u^2 + \xi - \frac{2\,\mu^2\xi}{x} \approx 2\,\mu^2 \ . \tag{5.14}$$

If we use these approximate equations, we see easily that the formula (5.09) gives just the approximate form of the equation of reflection (5.10).

Thus, the formulas (5.01) and (5.04), which were derived for the region near the shadow cone, reduce to those formulas which hold in the regions adjoining this shadow cone on both sides, and which were derived in our previous pages.

In conclusion, we make the following remarks about the formulas derived here.

In the same manner as the starting formulas for V, also the approximate formulas admit of transition to the case of a plane wave at suitable change of the expressions for the phase of the incident wave. This transition consists in letting x and $\sqrt{y_2}$ tend to infinity, while keeping their difference finite. As was shown in our papers [2] and [3], our starting formulas are valid in the case of a plane wave not only for a sphere, but also for a body of arbitrary shape. Therefore, we may regard the approximation formulas derived here, which contain the Fresnel integrals as special cases, as proven also for a body of arbitrary shape. It is also very probable that the diffraction pattern found here (a Fresnel diffraction, superimposed on a background) is valid, at least qualitatively, also in large distances from the body. One may, therefore, expect that the intensity of the background decreases with increasing distance from the body.

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X. FRESNEL REFLECTION LAWS AND DIFFRACTION LAWS

V. A. Fock

In 1821, the French scientist Fresnel established formulas determining the intensity and direction of oscillations in reflected and refracted rays of light incident on the plane surface of a transparent body.

Fresnel obtained his formulas on the basis of the elastic theory of light under the assumption of the transverse oscillations of the elastic medium (ether) where he was obliged to introduce special hypotheses on the elasticity and density of the ether in media which differed from each other by the index of refraction. This derivation does not correspond with the modern view on the nature of light and has only historical interest at the present time. However, the formulas themselves were justified brilliantly by experiment and, later, as touchstones for the verification of the whole new theory of light.

In 1865, the electromagnetic theory of light, created by Maxwell, appeared, which would sustain this verification and, moreover, would give an explanation of an unusually wide circle of phenomena including those which were detected many years later such as: radiowaves (Hertz, Popov), light pressure (Lebedev) and many others.

The Fresnel reflection laws emerge from the Maxwell equations and the appropriate boundary conditions without any additional hypotheses, where it appears that the transverse oscillations analyzed by Fresnel must be understood as the oscillations of the electric vector.

The Fresnel laws are applicable not only to light but to electromagnetic oscillations of any frequency, including radiowaves. On the other hand, the Fresnel laws are generalized easily to the case when the waves fall on the plane surface of an absorbing body. The Fresnel formulas retain their form, with this sole difference, that the index of refraction n' must be replaced by a complex quantity; namely, the square root of the complex dielectric constant of the medium.

The Fresnel formulas permit the direct expression of the amplitude of the electromagnetic field of the reflected wave through the amplitude of the incident wave field, where their values on the reflecting surface are understood to be these amplitudes. If a plane wave falls on the surface, and if the reflecting surface itself is a plane, then the reflected-wave field amplitudes at a certain distance from the surface will be the same as on the surface itself; only the phase will depend on the distance from the surface. If the reflecting surface is convex, then the incident, parallel beam of rays becomes divergent after reflection. In such a case, when calculating the reflected-wave amplitudes at a given distance from the point where the reflection would occur, it is necessary to introduce a correction factor into the amplitude which would take into account the beam spreading after reflection. This factor can be found from purely geometric considerations.

The electromagnetic-wave reflection laws are very simple and convenient for the approximate formulation in Fresnel formulas. It is a much less satisfactory matter in the case of the approximate formulation of the diffraction laws; i.e., the enveloping of an obstacle by the wave and its entrance into the geometric shade region. All the known until very recently approximate methods refer to the case of wave diffraction from an obstacele with sharp edges, for example, from an opaque screen with orifices. Basically, these methods are refinements of the Huygens principle. The principal step in this direction was made by Fresnel himself. According to the Huygens principle in the Fresnel formulation, part of the light wave covered by the screen does not act at all, but the uncovered regions act just as though there were no screen at all. A further improvement was made in 1882 by Kirchhoff who proposed a formula for the amplitude of light waves outside a screen. The Kirchhoff formula is a very flexible and convenient means of solving approximately the problem of diffraction from a screen with sharp edges but it does not take into account the influence of the screen material and, in general, does not take the limit conditions for the field which result from the Maxwell equations, into account.

The next substantial step in the solution of the diffraction problem from a screen with sharp edges is related to the finding of rigorous solutions to the Maxwell equations for certain particular cases (half-plane, wedge). Here the work of Sommerfeld should be mentioned and also the work of S. L. Sobolev and V. I. Smirnov, who approached the problem from a new point of view (nonstationary process). The extremely interesting problems of the plane and cylindrical waveguides with open ends (where the diffracted wave can be sent backward) were solved recently by the young Soviet scientist L. A. Vainshtein.

In contrast to the problem of diffraction from bodies with sharp edges (screens and diaphragms), no general approximate methods or approximate formulas (similar to the Kirchhoff formulas) have been proposed to solve the problem of diffraction from bodies with continuously varying curvature, to the present time. In order to find the field obtainable because of the diffraction of the incident wave, it was proposed to solve the Maxwell equations with the limiting conditions for each separate case, which is a very complex mathematical problem.

The Fresnel reflection formulas are integral laws in the sense that their use does not require the solution of the differential equations because these formulas give explicit expressions for the reflected wave amplitudes. Not only was the form of the appropriate integral law not known for the phenomenon of diffraction from bodies of arbitrary shape, but the fact of the existence of such a law was not established. In other words, the possibility was not established of writing explicit expressions for the field amplitude of waves bending around a body under any general assumptions of the electrical properties of the body material and on the shape of its surface.

To a known degree, this gap was filled in our works on the diffraction of plane waves from the surface of a convex, conducting body of arbitrary shape.

The assumption that the body material is a good conductor is essential because it affords the possibility of using the simplified, boundary conditions for a field which M. A. Leontovich established.

Considering the field near the body surface (at distances which are small in comparison with the radius of curvature of the surface), we established that this field has local character in the penumbra region. This means that the field in the penumbra region for a given incident wavelength, amplitude and polarization, depends only on the shape and properties of the body near the given point, where it is expressed through certain universal functions which can be tablulated once and for all. Hence, it appears to be possible to formulate certain general diffraction laws thereby.

Our formulas for the field can be considered as a generalization of the Fresnel formulas - a generalization which includes both the reflection and the diffraction laws.

Let us move, mentally, along the surface of a body from its illuminated side to the shade. The incident and reflected waves can be differentiated on the illuminated side, where the latter will be described well by the Fresnel formula. Near the geometric boundary of the shade, in the region of oblique incident of the ray, both waves are already inseparable from each other so that only consideration of the resultant field has meaning. Here, our formulas become valid while the Fresnel formulas become inapplicable. We do not have waves of more or less constant amplitude beyond the geometric shade boundaries but we have a damping wave, i.e., a wave with amplitude decreasing exponentially as the distance from the geometric shade boundaries increases. Here, the diffraction phenomenon occurs in its proper sense, where the diffraction law is transformed by our formulas.

From the above, it is clear that a region exists (namely, the region of oblique ray incidence) where both our diffraction formulas and the Fresnel formulas are correct simultaneously. Evidently, one formula must transform into the other in this region.

Later, we will cite the Fresnel formulas for an electromagnetic field and we will give their generalization which permits taking into account the broadening of the beam after it is reflected from a convex body. Furthermore, we will write the diffraction formulas we obtained, we will analyze their limiting case and we will trace how they transform into the Fresnel formulas in the region of oblique ray incidence.

1. Fresnel Reflection Laws

Let us denote the amplitudes of the electric and magnetic vectors of an incident wave at a given point of the body surface through $\vec{E}^\circ(E_x^\circ, E_y^\circ, E_z^\circ)$ and $\vec{H}_x^\circ, H_y^\circ, H_z^\circ)$. Let us denote the corresponding quantities for the reflected wave through $\vec{E}^*(E_x^*, E_y^*, E_z^*)$ and $\vec{H}^*(H_x^*, H_y^*, H_z^*)$. Furthermore, let $\vec{a}(a_x, a_y, a_z)$ be the unit vector in the incident ray direction and let $\vec{a}^*(a_x^*, a_y^*, a_z^*)$ be the unit vector in the reflected ray direction and let $\vec{n}(n_x, n_y, n_z)$ be the unit vector normal to the body surface at the incident point. According to the reflection law, the $\vec{a}^*$, $\vec{a}$ and $\vec{n}$ are related thus:

$$\vec{a}^* - \vec{a} = 2\vec{n}(\vec{a}\cdot\vec{n}) \qquad (1.01)$$

where

$$\vec{a}^{*}\cdot\vec{n} = -\vec{a}\cdot\vec{n} = \cos\theta, \qquad (1.02)$$

where θ is the incident angle. The a and a^* are proportional to the gradient of the phases of the incident and reflected waves. Considering the amplitude to be a quantity which varies slowly in comparison with the phase, we obtain from the Maxwell equations for a vacuum:

$$[a \times E^\circ] = H^\circ; \quad a\cdot E^\circ = 0, \qquad (1.03)$$

from which

$$[a \times H^\circ] = -E^\circ; \quad a\cdot H^\circ = 0, \qquad (1.04)$$

and similarly for the reflected wave:

$$[a^* \times E^*] = H^*; \qquad a^*\cdot E^* = 0, \qquad (1.05)$$

$$[a^* \times H^*] = -E^*; \qquad a^*\cdot H^* = 0. \qquad (1.06)$$

Let us denote the magnetic permeability through μ, the complex dielectric constant of the substance of the reflecting body through:

$$\eta = \epsilon + i\,\frac{4\pi\sigma}{\omega} \qquad (1.07)$$

and let us introduce the Fresnel coefficients:

$$N = \frac{\eta\cos\theta - \sqrt{\mu\eta - \sin^2\theta}}{\eta\cos\theta + \sqrt{\mu\eta - \sin^2\theta}}, \qquad (1.08)$$

$$M = \frac{\mu\cos\theta - \sqrt{\mu\eta - \sin^2\theta}}{\mu\cos\theta + \sqrt{\mu\eta - \sin^2\theta}}. \qquad (1.09)$$

Then the Fresnel formulas establishing the relation between the amplitudes of the incident and reflected waves can be written thus:

$$(n \cdot E^{*}) = N(n \cdot E^{\circ}) \tag{1.10}$$

$$(n \cdot H^{*}) = M(n \cdot H^{\circ}) \tag{1.11}$$

The amplitudes of the transmitted waves (penetrating the substance of the body) are not of interest and we will not write the corresponding formulas.

Equations (1.05), (1.10) and (1.11) can be solved with respect to the E^{*} and H^{*} vectors. Introducing the notation:

$$n \cdot E^{\circ} = E_n^{\circ} \; ; \quad n \cdot H^{\circ} = H_n^{\circ} \tag{1.12}$$

and expressing a^{*} through a according to (1.01), we will have:

$$\sin^2 \theta E^{*} = - NE_n^{\circ}(n \cos 2\theta + a \cos \theta) + MH_n^{\circ} \left[n \times a \right], \tag{1.13}$$

$$\sin^2 \theta H^{*} = - MH_n^{\circ}(n \cos 2\theta + a \cos \theta) - NE_n^{\circ} \left[n \times a \right]. \tag{1.14}$$

Such are the amplitudes of the reflected waves on the body surface which result from the Fresnel formulas.

Relations for the total field can also be derived from the preceding formulas. Denoting the total field on the body surface through E and H and their normal components through E_n and H_n and assuming:

$$\chi = \sqrt{1 - \frac{\sin^2\theta}{\eta\mu}}\ , \tag{1.15}$$

we will have:

$$\sin^2 \theta(E - nE_n) = \chi \sqrt{\frac{\mu}{\eta}}\ E_n \left\{ a - n(a \cdot n) \right\} + H_n \left[n \times a \right], \tag{1.16}$$

$$\sin^2\theta\,[n \times H] = E_n\left\{a - n(a\cdot n)\right\} + \chi\sqrt{\frac{\eta}{\mu}}\,H_n\,[n \times a]\,. \tag{1.17}$$

If $|\eta\mu| >> 1$, then $\chi = 1$ approximately and the right side of (1.16) and (1.17) are mutually proportional. In this case:

$$E - nE_n = \sqrt{\frac{\mu}{\eta}}\,[n \times H]\,. \tag{1.18}$$

The last relation already does not contain the vector a, i.e., is independent of the incident wave direction. As shown by M. A. Leontovich, it holds not only in the illuminated regio where the Fresnel formulas are applicable but on the whole body surface.

The following relations can also be derived from (1.16) and (1.17):

$$(a\cdot E) = \left(-\cos\theta + \chi\sqrt{\frac{\mu}{\eta}}\right)E_n\,, \tag{1.19}$$

$$(a\cdot H) = \left(-\cos\theta + \chi\sqrt{\frac{\eta}{\mu}}\right)H_n\,. \tag{1.20}$$

If the incident wave is plane so that the vector a has a specific value, then the last relations can be used instead of the Leontovich conditions (1.18). This is convenient when oblique ray incidence is considered where $\sin^2\theta = 1$ can be substituted for χ in (1.15).

2. Cross-Section of a Beam of Reflected Rays

In order to find the amplitude of the reflected waves at a certain distance from the body surface, it is necessary to have formulas for the cross-section of a beam resting on the dS area of the body surface, having traversed the given path s after reflection. These formulas can be derived from well-known formulas of differential geometry.

Let the equations of the reflecting surface be:

$$x = x_o(u,v); \quad y = y_o(u,v); \quad z = z_o(u,v) , \tag{2.01}$$

where u,v are the Gaussian coordinate parameters. The square of the element of arc on the surface can be written thus:

$$d\ell^2 = g_{uu}du^2 + 2g_{uv}du\,dv + g_{vv}dv^2 = \sum_{u,v} g_{uv}du\,dv , \tag{2.02}$$

where the sum $\sum_{u,v}$ is a shorthand notation for the middle term of this equality.

We will use notations for the covariant and contravariant components of the vectors and tensors, by raising and lowering the signs using the 'metric' tensor which enters into (2.02). We will write the surface element thus:

$$dS = \sqrt{g}\,du\,dv . \tag{2.03}$$

Let us write the formulas for the vector components normal to the surface and for their derivatives with respect to u,v. We will have:

$$\sqrt{g}\,n_x = \frac{\partial y_o}{\partial u}\frac{\partial z_o}{\partial v} - \frac{\partial y_o}{\partial v}\frac{\partial z_o}{\partial u} \quad \text{etc.} \tag{2.04}$$

$$\frac{\partial n_x}{\partial u} = -\sum_v G_u^v \frac{\partial x_o}{\partial v} \quad \text{etc.} \tag{2.05}$$

The last formula can be used to define G_u^v - the mixed component of the second quadratic form of the surface. If R_1 and R_2 are the principal radii of curvature of the normal cross-section of the surface, then we will have:

$$K = \frac{1}{R_1 R_2} = G_u^u G_v^v - G_v^u G_u^v , \qquad (2.06)$$

$$\frac{1}{R_1} + \frac{1}{R_2} = - G = - G_u^u - G_v^v . \qquad (2.07)$$

The quantity K is the Gaussian curvature of the surface. We will require the formula for the R_o radius of curvature of a normal cross-section of the surface by the plane of the incident ray. It can be shown that if $k\psi$ is the phase of the incident wave, where

$$(\text{grad } \psi)^2 = 1 , \qquad (2.08)$$

then

$$\sum_{u,v} g^{uv} \frac{\partial \psi_o}{\partial u} \frac{\partial \psi_o}{\partial v} = \sin^2 \theta , \qquad (2.09)$$

where θ is the incident angle and the derivatives are taken at the $\psi = \psi_o$ values of the phase of the body surface. The quantity R_o is then determined from the equality:

$$\sum_{u,v} G^{uv} \frac{\partial \psi_o}{\partial u} \frac{\partial \psi_o}{\partial v} = - \frac{\sin^2 \theta}{R_o} . \qquad (2.10)$$

Let us use the formulas written here to calculate the normal cross-section of a beam of rays reflected from the dS surface element.

Let us consider the equations:

$$\begin{aligned} x &= x_o + s a_x^* , \\ y &= y_o + s a_y^* , \qquad (2.11) \\ z &= z_o + s a_z^* , \end{aligned}$$

in which s is a certain given quantity and $x_o, y_o, z_o, a_x^*, a_y^*, a_z^*$ are functions of u,v determined from the equation (2.01) of the surface and from the relation:

$$a^* = a - 2n(a \cdot n) \; , \tag{2.12}$$

where n is the normal vector at x_o, y_o, z_o .

Evidently, s is the path traversed by the beam after reflection. For constant s, (2.11) are the equations of a certain surface parallel, in a known sense, to the reflecting body surface. If we were to vary u,v between (u,u + du), (v,v + dv), we would obtain a certain section of the surface (2.11). This section can be considered as the cross-section of a beam of reflected rays resting on the element of the surface $dS = \sqrt{g}\, du\, dv$. In order to obtain a normal beam cross-section, we must project this section onto a plane perpendicular to the reflected ray. Denoting the area of the normal section through D(s)dS, we will have

$$D(s)dS = \begin{vmatrix} a_x^* & a_y^* & a_z^* \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix} du\, dv \; , \tag{2.13}$$

from which:

$$D(s) = \frac{1}{\sqrt{g}} \begin{vmatrix} a_x^* & a_y^* & a_z^* \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix} . \tag{2.14}$$

We calculate this determinant under the assumption that the incident wave is planar and that, therefore, the vector a is independent of u,v.

After sufficiently complex computations, which we omit here, the following result is obtained:

$$D(s) = \cos\theta + 2s\left(-G + G\sum_{u,v} g^{uv}\frac{\partial\psi_o}{\partial u}\frac{\partial\psi_o}{\partial v} - \sum_{uv} G^{uv}\frac{\partial\psi_o}{\partial u}\frac{\partial\psi_o}{\partial v}\right) + 4Ks^2\cos\theta\,. \tag{2.15}$$

Using (2.06) - (2.10) cited above, we can write:

$$D(s) = \cos\theta + 2s\left[\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\cos^2\theta + \frac{\sin^2\theta}{R_o}\right] + \frac{4s^2}{R_1R_2}\cos\theta\,, \tag{2.16}$$

where the values of R_o, R_1, R_2 are taken at the point where reflection occurred.

Evidently, $\frac{D(s)}{D(0)}$ yields the beam broadening, i.e., the ratio of this cross-section at the distance s from the surface (we measure along the ray) to the cross-section at the surface itself.

3. Electromagnetic Field of the Reflected Wave

Let the field of the incident plane wave equal:

$$E^o e^{i\psi}\,,\quad H^o e^{i\psi}\,, \tag{3.01}$$

where E^o and H^o are constant amplitudes and

$$\phi = k\psi = k(xa_x + ya_y + za_z) \tag{3.02}$$

is the phase of the wave at this space point.

Introducing the value

$$\phi_o = k\psi_o = k(x_o a_x + y_o a_y + z_o a_z) \qquad (3.03)$$

of the phase ϕ on the body surface, we will have for the incident wave field on the body surface:

$$E^o e^{ik\psi_o}, \quad H^o e^{ik\psi_o}. \qquad (3.04)$$

The reflected wave field on the body surface will equal:

$$E^* e^{ik\psi_o}, \quad H^* e^{ik\psi_o}, \qquad (3.05)$$

where E^* and H^* are related to E^o and H^o through the Fresnel formulas (1.13) and (1.14). [Let us note, apropos of the notation, that in (1.13) and (1.14), we considered the phase factor $e^{ik\psi_o}$ to be included in E^o, H^o and in E^*, H^*, but since this factor is identical in both sides of (1.13) and (1.14) then it does not matter whether we understand the total expressions (3.04) and (3.05) in these equalities or their amplitudes.]

In the notation of this paragraph, E^o and H^o are constants and E^* and H^* are slowly varying functions of the coordinate point on the surface.

Let us denote one of the reflected wave field components through F. The value of F on the surface will be:

$$F = f(u,v)\, e^{ik\psi_o(u,v)}, \qquad (3.06)$$

where $f(u,v)$ is a slowly varying function and k is a large parameter. In order to find F at a certain distance s from the surface, we must know the solution of the wave equation:

$$\Delta F + k^2 F = 0, \qquad (3.07)$$

which satisfies the radiation condition and the limit condition (3.06) on the surface. Treating k as a large parameter, the approximate form of such a solution can be shown explicitly.

Actually, let us consider the expression:

$$F = f(u,v)\sqrt{\frac{D(0)}{D(s)}}\cdot e^{ik(\psi_o+s)} \quad . \tag{3.08}$$

The u,v,s quantities can be interpreted as curvilinear coordinates of a space point related to the x,y,z rectangular coordinates through (2.11). The geometric meaning of these curvilinear coordinates is evident: the u,v parameters determine the position of that point on the body surface from which the ray, coming from x,y,z, is reflected; the quantity s is the distance traversed by the ray after reflection.

Therefore, F in (3.08) can be interpreted as a function of the space point. It is evident that this function takes the value (3.06) on the surface. It is also evident that it satisfies the radiation condition and it corresponds to a scattered wave. But, moreover, if the parameter k is large, then F satisfies the wave equation approximately. Actually, it can be shown that the equalities:

$$\left\{\mathrm{grad}(\psi_o + s)\right\}^2 = 1 \ , \tag{3.09}$$

$$\mathrm{div}\left\{f^2 \frac{D(0)}{D(s)}\,\mathrm{grad}(\psi_o + s)\right\} = 0 \ . \tag{3.10}$$

result from the definitions of ψ_o and D(s) and from (2.11).

On the basis of these equalities, it is easy to verify that second and first power terms in k drop out of (3.07) after F is substituted therein and only zero degree terms remain.

The correctness of (3.08) results, independently of the reasoning just explained, from geometric optics considerations. Actually, this expression must give the reflected wave. But, evidently, the phase of the reflected wave equals $k(\psi_o + s)$. As regards the amplitude, then, if we travel along a fine beam of reflected rays, the amplitude must vary in inverse proportion to the square root of the beam cross-section, as is given by (3.08).

Therefore, this formula gives the reflected wave field at the distance s from the surface when the field on the surface itself is known.

Applying this formula to the electric and magnetic field components, we obtain:

$$E = E^{*}(u,v) \cdot \sqrt{\frac{D(0)}{D(s)}}\; e^{ik(\psi_0 + s)}, \tag{3.11}$$

$$H = H^{*}(u,v) \cdot \sqrt{\frac{D(0)}{D(s)}}\; e^{ik(\psi_0 + s)}, \tag{3.12}$$

where $E^{*}(u,v)$ and $H^{*}(u,v)$ are the field amplitudes on the body surface obtained from the Fresnel formulas.

The formulas we obtained for the field are natural combinations of the reflection and geometric (ray) optics laws. Both, separately, were known over a hundred years ago: Fresnel found his reflection laws about 1820 and Hamilton found the ray optics laws about 1830. In particular, Hamilton knew that the quantity, corresponding to our D(s), is a second degree polynomial in s. However, we have not been able to find any indication, in the literature, of the application of these results to the approximate representation of reflected electromagnetic waves.

4. Diffraction Laws in the Penumbra Region

In the introduction, we already mentioned that the incident and reflected waves become mutually inseparable near the geometric shade boundaries, in the region of oblique ray incidence, and the Fresnel formulas become inapplicable. We explain here, on the basis of our work,[1] the idea of the derivation of the diffraction formulas which give the field in this region and also in the penumbra and umbra regions.

Let us visualize a convex body on which a plane wave falls in the x direction. Let us select a point on the body surface which lies on the boundary of geometric shade and let us make it the origin. Let us direct the z axis along the normal to the surface (towards the air). Since the normal on the shade boundary is perpendicular to the wave direction, then our x and z axes will be mutually perpendicular. We select the y axis so that we will obtain a right-handed coordinate system.

The equation of the surface in the neighborhood of this point will be:

$$z + \frac{1}{2}\left(ax^2 + 2bxy + cy^2\right) = 0 , \tag{4.01}$$

in which

$$a > 0 ; \quad c > 0 ; \quad ac - b^2 \geq 0 . \tag{4.02}$$

The radius of curvature of the normal cross-section of the surface will equal:

$$R_o = \frac{1}{a} . \tag{4.03}$$

Later, we will introduce the 'large parameter' m according to the formula:

$$m = \sqrt[3]{\frac{kR_o}{2}} = \sqrt[3]{\frac{k}{2a}} \tag{4.04}$$

and we will solve our problem by neglecting quantities of order $\frac{1}{m^2}$ in comparison with unity.

Our idea is to find the electromagnetic field at a small, compared to the radius of curvature R_o, distance from the origin.

Under our assumptions, each field component will be:

$$F = e^{ikx} F^*, \tag{4.05}$$

where F^* satisfies the differential equation:

$$\frac{\partial^2 F^*}{\partial z^2} + 2ik\frac{\partial F^*}{\partial x} = 0 . \tag{4.06}$$

All the field components can be expressed through H_y and H_z thus:

$$\left.\begin{aligned} E_x &= \frac{1}{k}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right), \\ E_y &= H_z , \\ E_z &= - H_y , \\ H_x &= \frac{1}{k}\left(\frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}\right), \end{aligned}\right\} \tag{4.07}$$

which can be considered as simplified Maxwell equations.

M. A. Leontovich established the approximate limit conditions for a field in air on the boundary of a good-conducting body. They are correct under the conditions:

$$|\eta\mu| \gg 1 ; \quad kR_o \left|\sqrt{\eta\mu}\right| \gg 1 \tag{4.08}$$

and have the form [see (1.18)]:

$$E - nE_n = \sqrt{\frac{\mu}{\eta}}\,[n \times H] . \tag{4.09}$$

Later, we will consider $\mu = 1$. The normal vector component in (4.09) is determined from the equation (4.01) of the surface. We can put, approximately:

$$n_x = ax + by ; \quad n_y = bx + cy ; \quad n_z = 1 , \tag{4.10}$$

because the squares of n_x and n_y can be neglected in comparison with unity. We will consider the quantities n_x, n_y, $\frac{1}{m}$, $\frac{1}{\sqrt{\eta}}$ to be small of one order.

Under these assumptions, the limit conditions for fields which contain only H_y and H_z can also be derived from (4.07) and (4.09). They will be:

$$H_z = - n_y H_y , \tag{4.11}$$

$$\frac{\partial H_y}{\partial z} + ik\left(n_x + \frac{1}{\sqrt{\eta}}\right) H_y = n_y \frac{\partial H_z}{\partial x} . \tag{4.12}$$

Because of the smallness of n_y, the right sides of these equations are correction terms. In a first approximation, they can be replaced by zero and the simpler limit conditions can be analyzed:

$$H_z = 0 , \tag{4.13}$$

$$\frac{\partial H_y}{\partial z} + ik\left(n_x + \frac{1}{\sqrt{\eta}}\right) H_y = 0 . \tag{4.14}$$

In the second approximation, H_y and H_z values obtained by solving the differential equations with the limit conditions (4.13) and (4.14)* can be substituted into the right sides of (4.11) and (4.12).

The solution must satisfy the conditions at infinity as well as the differential equation and the limit conditions. These former are the requirement that the part of the solution which corresponds to the plane wave should have a given amplitude at infinity.

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*In our basic work[1], an inconsistency was admitted here, namely, (4.11) and (4.14) were considered as the limit conditions. Consequently, both the principal and the correction term were obtained in the final expression for H_z [see (4.30) below] and only the principal term in the expression for H_y (4.29). This inaccuracy is corrected in this work.

The mathematical problem formulated has a unique solution which we will give here minus all the computations and being limited to definitions.

If we do not consider the e^{ikx} factor, the field will depend on the coordinates only through the quantities:*

$$\xi = m\,(ax + by)\ , \qquad (4.15)$$

$$\zeta = 2am^2 \left[z + \tfrac{1}{2}\,(ax^2 + 2bxy + cz^2\,) \right] , \qquad (4.16)$$

of which the second becomes zero on the surface. The constants, characterizing the electric properties of the reflecting surface, enter into the field expression through the quantities:**

$$q = \frac{im}{\sqrt{\eta}}\ ; \quad m = \sqrt[3]{\frac{k}{2a}}\ . \qquad (4.17)$$

All things considered, the field is expressed through one universal (i.e., independent of the surface shape) function $V_1(\xi,\zeta,q)$ and through its limit value:

$$V_2\,(\xi,\zeta) = V_1\,(\xi,\zeta,\infty)\ . \qquad (4.18)$$

The V_1 function can be represented as a definite integral containing the complex Airy functions $w_1(t)$ and $w_2(t)$. These latter are defined as the solution of the differential equation:

$$w''(t) = tw(t)\ , \qquad (4.19)$$

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*Moreover, the correction terms will contain y in a linear way.

**If we should use (1.19) and (1.20) instead of the Leontovich conditions, we would obtain for q the somewhat more exact expression: $q = \frac{im}{\eta}\sqrt{\eta - 1}$.

which have, for large negative t, the asymptotic expansions:

$$w_1(t) = \frac{1}{\sqrt[4]{-t}} \exp\left(i\,\frac{2}{3}(-t)^{3/2} + i\,\frac{\pi}{4}\right), \qquad (4.20)$$

$$w_2(t) = \frac{1}{\sqrt[4]{-t}} \exp\left(-i\,\frac{2}{3}(-t)^{3/2} - i\,\frac{\pi}{4}\right). \qquad (4.21)$$

The expression for V_1 has the form:

$$V_1(\xi,\zeta,q) = \frac{i}{2\sqrt{\pi}} \int_C e^{i\xi t}\left\{w_2(t-\zeta) - \frac{w_2'(t)-qw_2(t)}{w_1'(t)-qw_1(t)}\, w_1(t-\zeta)\right\}dt\,, \qquad (4.22)$$

where the C contour goes along the ray arc $t = \frac{2}{3}\pi$ from infinity to zero and along the ray arc $t = -\frac{1}{3}\pi$ from zero to infinity.

For $\zeta = 0$ (on the body surface) the V_1 expression simplifies and becomes:

$$V_1(\xi,0,q) = \frac{1}{\sqrt{\pi}} \int_C e^{i\xi t}\, \frac{dt}{w_1'(t) - qw_1(t)}\,. \qquad (4.23)$$

This function is tabulated for a number of values of q; the tables for q = 0 (absolutely-conducting body) are printed in our work.[2]

Having the definition of $V_1(\xi,\zeta,q)$, we can write the expression for the field. To do this, let us introduce the functions:

$$\Psi = e^{-i\phi}\, V_1(\xi,\zeta,q)\,, \qquad (4.24)$$

$$\Phi = e^{-i\phi}\, V_2(\xi,\zeta)\,, \qquad (4.25)$$

where

$$\phi = \xi\zeta - \frac{1}{3}\xi^3 , \tag{4.26}$$

and let us form the following expressions with their aid:

$$P = - i \frac{b}{a} \frac{\partial \Psi}{\partial \zeta} + \left(i \frac{b}{a} q + \frac{ac-b^2}{a} my \right) (\Phi - \Psi) , \tag{4.27}$$

$$Q = i \frac{b}{a} \frac{\partial \Phi}{\partial \zeta} + \left(- i \frac{b}{a} q + \frac{ac-b^2}{a} my \right) (\Phi - \Psi) , \tag{4.28}$$

Then the H_y and H_z magnetic field components will equal:

$$H_y = H_y^o e^{ikx} \Psi + \frac{1}{m} H_z^o e^{ikx} Q , \tag{4.29}$$

$$H_z = \frac{1}{m} H_y^o e^{ikx} P + H_z^o e^{ikx} \Phi , \tag{4.30}$$

where H_y^o and H_z^o are the amplitudes of the incident wave. All four of the functions, Φ, Ψ, P, Q, satisfy the differential equation of (4.06) type and will be of the same order of magnitude. Since m is a large parameter, then the terms containing Φ and Ψ will be the principal, and the term containing P and Q will be corrective. The E_x and H_x field components will be of the same order as the correction terms, namely:

$$E_x = - \frac{i}{m} H_y^o e^{ikx} \frac{\partial \Psi}{\partial \zeta} , \tag{4.31}$$

$$H_x = \frac{i}{m} H_z^o e^{ikx} \frac{\partial \Phi}{\partial \zeta} . \tag{4.32}$$

As regards the remaining electric field components, they will equal:

$$E_y = H_z ; \quad E_z = - H_y . \tag{4.33}$$

because of the simplified Maxwell equations (4.07).

Hence, we have determined all the field components.

5. Investigation of the Expressions for the Fields in the Umbra and Direct-Visibility Regions

The diffraction formulas we derived, give the field near a certain point on the surface of a conducting body in the geometric umbra boundary. We show that they give a continuous transition from the field corresponding to the Fresnel formulas (for the direct-visibility region) to total shadow. Let us start with the umbra region.

The integral (4.22) can be represented as the sum of residues referred to the roots of the denominator of the integrand. We have:

$$V_1(\xi,\zeta,q) = i\, 2\sqrt{\pi} \sum_{s=1}^{\infty} e^{i\xi t_s} \frac{w_1(t_s - \zeta)}{w_1^2(t_s)\,(t_s - q^2)}\,, \quad (5.01)$$

where t_s is a root of the equation:

$$w_1'(t_s) - q w_1(t_s) = 0\,. \quad (5.02)$$

The t_s roots lie near the ray arc $t = \frac{\pi}{3}$ and increase in absolute value. For sufficiently large positive values of $\xi - \sqrt{\zeta}$, we can be limited to one term in the series of (5.01). Moreover, if the asymptotic expansion (4.20) for w_1 is used and if ζ is considered to be large in comparison with t_1 in it, then we obtain the approximate expression for V_1:

$$V_1(\xi,\zeta,q) = \frac{e^{i\frac{3\pi}{4}}\; 2\sqrt{\pi}}{w_1^2(t_1)\,(t_1 - q^2)} \cdot e^{i\frac{2}{3}\zeta^{3/2}} \cdot e^{i(\xi - \sqrt{\zeta})\,t_1}\,. \quad (5.03)$$

The quantity t_1 has the following values for $q = 0$ and $q = \infty$:

$$t_1 = 1{,}01879 \cdot e^{i\frac{\pi}{3}} \qquad (q = 0)\ , \tag{5.04}$$

$$t_1 = 2{,}33811 \cdot e^{i\frac{\pi}{3}} \qquad (q = \infty). \tag{5.05}$$

In every case, both the real and the imaginary parts of t_1 are positive. Hence, there follows that the V_1 and V_2 are the Φ, Ψ, P and Q functions related to them and, therefore, the field, will decrease exponentially as $\xi - \sqrt{\zeta}$ increases.

Let us note that the equality $\xi - \sqrt{\zeta} = 0$ yields the geometric boundary of the umbra. The increasing, positive values of $\xi - \sqrt{\zeta}$ correspond to points lying farther and farther in the umbra region.

Where the magnitude of $\xi - \sqrt{\zeta}$ is small (it can be of either sign) we there have the penumbra. We will not dwell on methods of computing the V_1 function in this region; let us say only that this function and, therefore, the field varies continuously there.

Now, let us turn to the line-of-sight region where $\xi - \sqrt{\zeta}$ is large and negative. In this case, it is impossible to use the series (5.01) for V_1 and it is necessary to return to the (4.22) integral. The term containing the $w_2(t - \zeta)$ in this integral can be computed exactly. It yields:

$$\frac{1}{2\sqrt{\pi}} \int_C e^{i\xi t} \ w_2(t - \zeta)\ dt = e^{i\phi}\ , \tag{5.06}$$

where ϕ has the value:

$$\phi = \xi\zeta - \frac{1}{3}\ \xi^3\ , \tag{5.07}$$

which agrees with (4.26). Therefore, this term yields the component unity in the Φ and Ψ functions and it corresponds to an incident wave in the field expressions.

The second term can be evaluated according to the stationary phase method as shown in [1]. The phase extremum is obtained for $\sqrt{-t} = p$, where:

$$p = \frac{1}{3}\left(\sqrt{\xi^2 + 3\zeta} - 2\xi\right) . \tag{5.08}$$

It is convenient to introduce the special notation:

$$\sigma = \sqrt{\xi^2 + 3\zeta} . \tag{5.09}$$

for the square root in the above formula. Let us note that p has the same sign as $\sqrt{\zeta} - \xi$ since $p > 0$ corresponds to the line-of-sight region, $p = 0$ to the geometric boundary of the umbra and $p < 0$ to the umbra. We are interested now in large positive values of p. Use of the stationary phase method for this case yields for all the V_1:

$$V_1(\xi,\zeta,q) = e^{i\phi} - e^{i\phi^*} \cdot \sqrt{\frac{p}{\sigma}} \cdot \frac{q - ip}{q + ip} , \tag{5.10}$$

where the phase ϕ equals (5.07) and the phase ϕ^* equals:

$$\phi^* = \frac{1}{27}\left(4\sigma^3 - 3\sigma^2\xi - 2\xi^3\right) . \tag{5.11}$$

Let us note that the phase difference $\phi^* - \phi$ equals:

$$\phi^* - \phi = \frac{2}{27}(\sigma + \xi)(\sigma - 2\xi)^2 = (\sigma - p)\,p^2 . \tag{5.12}$$

Since $\phi^* - \phi$ goes to zero on the body surface, $\sigma = p = -\xi$ for $\zeta = 0$.

The quantity V_2 is obtained from (5.10) for $q = \infty$. The Φ and Ψ functions related to V_1 and V_2 will equal approximately:

$$\Psi = 1 - e^{i(\phi^* - \phi)} \sqrt{\frac{p}{\sigma}} \cdot \frac{q - ip}{q + ip} , \qquad (5.13)$$

$$\Phi = 1 - e^{i(\phi^* - \phi)} \sqrt{\frac{p}{\sigma}} . \qquad (5.14)$$

Not only the functions Ψ and Φ themselves enter into the field expression but also their derivatives with respect to ζ. All the factors, except the phase, can be considered constant when forming the derivatives. Because:

$$\frac{\partial(\phi^* - \phi)}{\partial\zeta} = \frac{2}{3}\sigma - \frac{4}{3}\xi = 2p \qquad (5.15)$$

we will have:

$$\frac{\partial\Psi}{\partial\zeta} = 2ip\,(\Psi - 1) \; ; \quad \frac{\partial\Phi}{\partial\zeta} = 2ip\,(\Phi - 1) . \qquad (5.16)$$

Evaluating P and Q by using these values, we obtain:

$$P = Q = \frac{2ip}{q + ip}\left(\frac{b}{a}\,p - \frac{ac - b^2}{a}\,my\right) \sqrt{\frac{p}{\sigma}}\; e^{i(\phi^* - \phi)} . \qquad (5.17)$$

There remains but to substitute the expressions found into (4.29) - (4.32) for the field. Hence, it is convenient to denote the phase of the reflected wave by the one letter:

$$\chi = kx + \phi^* - \phi \qquad (5.18)$$

With this notation, we will have:

$$H_y = H_y^o\, e^{ikx} - H_y^o\, \frac{q - ip}{q + ip} \sqrt{\frac{p}{\sigma}}\; e^{i\chi} +$$

$$+ \frac{1}{m} H_z^o\, \frac{2ip}{q + ip}\left(\frac{b}{a}\,p - \frac{ac - b^2}{a}\,my\right) \sqrt{\frac{p}{\sigma}}\; e^{i\chi} , \qquad (5.19)$$

$$H_z = H_z^o e^{ikx} - H_z^o \sqrt{\frac{p}{\delta}}\, e^{i\chi} +$$

$$+ \frac{1}{m} H_y^o \frac{2ip}{q + ip} \left(\frac{b}{a} p - \frac{ac - b^2}{a} my \right) \sqrt{\frac{p}{\delta}}\, e^{i\chi} . \tag{5.20}$$

$$E_x = - \frac{1}{m} H_y^o \cdot 2p \frac{q - ip}{q + ip} \sqrt{\frac{p}{\delta}}\, e^{i\chi} , \tag{5.21}$$

$$H_x = \frac{1}{m} H_z^o \cdot 2p \sqrt{\frac{p}{\delta}}\, e^{i\chi} \tag{5.22}$$

and, moreover, $E_y = H_z$; $E_z = - H_y$.

The first terms in (5.19) and (5.20) evidently yield the incident wave and the remaining terms the reflected wave. In the next paragraph, we show that the reflected wave corresponds, in accuracy, to the Fresnel formula with a correction for beam broadening.

6. Comparison of the Diffraction Formula With the Fresnel Formula for the Line-of-Sight Region

Now, let us turn to the Fresnel formulas. Putting $\mu = 1$ in the Fresnel coefficients and considering $\sqrt{\eta}$ a large quantity and $\cos\theta$ to be small (of the order of $\frac{1}{\sqrt{\eta}}$, we obtain for N and M:

$$N = \frac{\sqrt{\eta}\cos\theta - 1}{\sqrt{\eta}\cos\theta + 1} ; \quad M = - 1 \tag{6.01}$$

We must put $a_x = 1$, $a_y = a_z = 0$ in the Fresnel formulas (1.13) and (1.14) and we must consider n_x and n_y small quantities for which the squares can be neglected. Then these formulas yield for the electric field:

$$\left.\begin{aligned} E_x^* &= -\,2Nn_x\,H_y^o\,, \\ E_y^* &= -\,H_z^o - (N+1)\,n_y\,H_y^o\,, \\ E_z^* &= -\,NH_y^o + (N+1)\,n_y\,H_z^o \end{aligned}\right\} \tag{6.02}$$

and for the magnetic field:

$$\left.\begin{aligned} H_x^* &= -\,2N_x\,H_z^o\,, \\ H_y^* &= NH_y^o - (N+1)\,n_y\,H_z^o\,, \\ H_z^* &= -\,H_z^o - (N+1)\,n_y\,H_y^o\,. \end{aligned}\right\} \tag{6.03}$$

In order ot obtain the reflected wave field at a certain distance from the surface, it is necessary, according to (3.11) and (3.12), to multiply these expressions by the factor:

$$\sqrt{\frac{D(0)}{D(s)}}\;e^{ik(x_o + s)}\,. \tag{6.04}$$

The value of these quantities, except s, must be taken at that x_o, y_o, z_o point where the reflection of the ray striking the x,y,z point, occurred. Since the equation of the reflecting surface is:

$$z_o + \frac{1}{2}\,(ax_o^2 + 2bx_oy_o + cy_o^2\,) = 0\,, \tag{6.05}$$

then we have:

$$n_x = ax_o + by_o\;;\quad n_y = bx_o + cy_o\;;\quad n_z = 1\,. \tag{6.06}$$

In evaluating D(s) according to the general formula (2.23), we must neglect the last term since we are interested in the field at distances which are small in comparison with the radius of curvature. The remaining terms yield:

$$D(s) = \cos\theta + 2as = 2as - ax_o - by_o \, . \qquad (6.07)$$

In order to make a comparison of the diffraction formulas (5.19) - (5.22) and the Fresnel formulas (6.02), (6.03), we must establish the relationship between the x_o, y_o, s quantities and the x,y,z coordinates (or the ξ, ζ, y quantities). This relationship is given by (2.11), which becomes in our case:

$$\left. \begin{aligned} x &= x_o + s - 2sn_{x2} \, , \\ y &= y_o - 2sn_x n_y \, , \\ z &= z_o - 2sn_x n_z \, . \end{aligned} \right\} \qquad (6.08)$$

Solving these equations, approximately, with respect to x_o, y_o, s, we obtain:

$$\left| \begin{aligned} ax_o + by_o &= \frac{2\xi - \sigma}{3m} = -\frac{p}{m} \, , \\ y_o &= y \, , \\ s &= \frac{\sigma + \xi}{3am} = \frac{\sigma - p}{2am} \, . \end{aligned} \right. \qquad (6.09)$$

Hence:

$$n_x = -\frac{p}{m} \, ; \quad n_y = -\frac{b}{a}\frac{p}{m} + \frac{ac - b^2}{a} y \, . \qquad (6.10)$$

Furthermore, according to (5.12), (5.13), the phase χ equals:

$$\chi = kx + \phi^* - \phi = kx + (\sigma - p)\, p^2 = k(x + 2sn_x^2) = k(x_o + s) \, , \qquad (6.11)$$

that is, it equals the phase of the reflected wave calculated according to geometric optics. Let us now calculate the magnitude of D(s). Substituting (6.09) into (6.07), we obtain:

$$D(s) = \frac{\sigma}{m} , \tag{6.12}$$

in which, evidently,

$$D(0) = \cos\theta = \frac{p}{m} . \tag{6.13}$$

The last three formulas yield:

$$\sqrt{\frac{p}{\sigma}}\; e^{i\chi} = \sqrt{\frac{D(0)}{D(s)}}\; e^{ik(x_o + s)} \tag{6.14}$$

Therefore, the factor (6.14) which enters into all the expressions for the reflected wave in the diffraction formulas (5.19) - (5.22) agrees with the factor which enters into (3.08) - (3.09) which are generalizations of the Fresnel formulas. The quantity:

$$\sqrt{\frac{p}{\sigma}} = \sqrt{\frac{1}{3} - \frac{2\xi}{3\sigma}} \tag{6.15}$$

here yields the correction for beam broadening.

There remains to verify that all the other quantities in (5.19) - (5.22) agree with the Fresnel.

According to (4.17) and (6.13), we have:

$$q = \frac{im}{\sqrt{\eta}} ; \quad p = m\cos\theta . \tag{6.16}$$

Consequently:

$$\frac{q - ip}{q + ip} = \frac{1 - \sqrt{\eta}\cos\theta}{1 + \sqrt{\eta}\cos\theta} = - N , \tag{6.17}$$

where N is the Fresnel coefficient (6.01).*

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*The value $q = \frac{im}{\eta}\sqrt{\eta - 1}$ leads to a rather more exact value of N, namely:

$$N = \frac{\eta\cos\theta - \sqrt{\eta - 1}}{\eta\cos\theta + \sqrt{\eta - 1}} .$$

Using (6.19) and (6.17) as notation, we can write (5.19) - (5.22) for the field thus:

$$H_y = H_y^o e^{ikx} + \left[NH_y^o - (N+1)\, n_y H_z^o\right] \sqrt{\frac{p}{\delta}}\, e^{i\chi}, \tag{6.18}$$

$$H_z = H_z^o e^{ikx} + \left[-H_z^o - (N+1)\, n_y H_y^o\right] \sqrt{\frac{p}{\delta}}\, e^{i\chi}, \tag{6.19}$$

$$E_x = -2\, Nn_x H_y^o \sqrt{\frac{p}{\delta}}\, e^{i\chi}, \tag{6.20}$$

$$H_x = -2n_x H_z^o \sqrt{\frac{p}{\delta}}\, e^{i\chi}. \tag{6.21}$$

Comparing these expressions with the Fresnel formulas (6.02) and (6.03) we state that the factors with the magnitude of (6.14) agree in accuracy with their Fresnel values H_y^*, H_z^*, E_x^*, H_x^*. The equalities $E_y = H_z$ and $E_z = -H_y$ are satisfied both in the case of our formulas and in the case of the Fresnel formulas.

Therefore, we showed that our formulas transform into the generalized [by the introduction of the (6.14) factor] Fresnel formulas in that part of the line-of-sight region where the slope of the angle made by the ray with surface of the body is small.

In the penumbra and umbra regions our formulas yield a diffraction picture.

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XI. GENERALIZATION OF THE REFLECTION FORMULAS TO THE CASE OF REFLECTION OF AN ARBITRARY WAVE FROM A SURFACE OF ARBITRARY FORM

V. A. Fock

ABSTRACT

Fresnel formulas and the laws of Hamilton's ray optics are used as a basis for derivation of expressions for an electromagnetic field of an arbitrary wave reflected from a surface of arbitrary form. A correction for the dilation of the pencil of rays after reflection is considered. In the derivation the tensor form of the differential geometry of the reflecting surface is used. The Gaussian parameters of the surface in the point of reflection and the phase of the reflected wave are considered as curvilinear coordinates. For the specific case of a spherical wave reflected from a sphere, the formulas obtained are compared with those obtained from diffraction theory.

In our paper "The laws of Fresnel reflection and the laws of diffraction" [subsequently referred to as (Ref. 1)], a reflection formula considering the cross section of the bundle of reflected rays was derived for the case of a plane wave reflected from a surface of an arbitrary form. This formula was then compared with diffraction formulas valid in the half-shadow region.

In the present work the reflection formula is derived for the case of reflection of an arbitrary (not a plane) wave. Our calculations are based on the application of the laws of Fresnel reflection, established by him around 1820, and the laws of ray optics established by Hamilton around 1830. Our results cannot therefore be considered as principally new. Inasmuch as the Fresnel formulas are applied by us to the electromagnetic field, however, and inasmuch as the laws of the ray optics are formulated by us with the aid of geometry in its present day tensor form (which leads to extraordinary large simplifications), our results may prove to be useful for a practical application. For the convenience of the reader unfamiliar with the tensor form of differential geometry, we present a compilation (in Sect. 2) of the necessary formulas.

1. Fresnel formulas

Let the field of an incident wave be represented by

$$\underset{\sim}{E}^{o} e^{ik\psi}, \quad \underset{\sim}{H}^{o} e^{ik\psi}, \tag{1.1}$$

where E^{o} and H^{o} denotes amplitude, and ψ is the phase expressed in units of length, and

$$(\text{grad } \psi)^2 = 1. \tag{1.2}$$

For a plane wave, the amplitudes $\underset{\sim}{E}^{o}$ and $\underset{\sim}{H}^{o}$ are constant; in the general case we shall consider the components of vectors $\underset{\sim}{E}^{o}$ and $\underset{\sim}{H}^{o}$ as slowly variable functions of coordinates. In the following, $\underset{\sim}{E}^{o}$ and $\underset{\sim}{H}^{o}$ are understood to be the values of the amplitude of the field on the surface of a reflecting body. The corresponding values for a reflected wave will be designated by E^{1} and $\underset{\sim}{H}^{1}$.

Let, furthermore, $\underset{\sim}{a}(a_x a_y a_z)$ be a single vector in the direction of an incident ray, $a^1(a^1_x a^1_y a^1_z)$ - a single vector in the direction of a reflected ray, and $n(n_x n_y n_z)$ - the single vector of a normal to the surface of the body in the point of reflection. According to the law of reflection, the values a^1, a and n are related by a relation:

$$a^1 = a - 2n(a \cdot n), \tag{1.3}$$

moreover

$$a^1 \cdot n = - a \cdot n = \cos\theta , \tag{1.4}$$

where θ is the angle of incidence. The values a and a^1 are proportional to the gradient of the phase of incident and reflected wave. Neglecting the variation of the amplitude over one wavelength, we obtain from the Maxwell equation for the vacuum

$$[a \times E^0] = H^0; \quad a \cdot E^0 = 0, \tag{1.5}$$

whence

$$[a \times H^0] = - E^0; \quad a \cdot H^0 = 0, \tag{1.6}$$

and analogously for the reflected wave

$$[a^1 \times E^1] = H^1; \quad a^1 \cdot E^1 = 0; \tag{1.7}$$
$$[a^1 \times H^1] = - E^1; \quad a^1 \cdot H^1 = 0. \tag{1.8}$$

We designate by μ the magnetic permeability, and by

$$\eta = \epsilon + i4\pi\sigma/\omega \tag{1.9}$$

the complex dielectric constant of the substance of the reflecting body, and introduce the Fresnel coefficients

$$N = \frac{\eta \cos\theta - \sqrt{\mu\eta - \sin^2\theta}}{\eta \cos\theta + \sqrt{\mu\eta - \sin^2\theta}}, \tag{1.10}$$

$$M = \frac{\mu \cos\theta - \sqrt{\mu\eta - \sin^2\theta}}{\mu \cos\theta + \sqrt{\mu\eta - \sin^2\theta}}. \tag{1.11}$$

Then Fresnel formulas, which define the relation between the amplitudes of the incident and reflected wave, can be written in the form:

$$(\underset{\sim}{n}\cdot\underset{\sim}{E}^1) = N(\underset{\sim}{n}\cdot\underset{\sim}{E}^o), \tag{1.12}$$

$$(\underset{\sim}{n}\cdot\underset{\sim}{H}^1) = M(\underset{\sim}{n}\cdot\underset{\sim}{H}^o). \tag{1.13}$$

The amplitudes of the transmitted wave (which penetrates the body of the substance) are of no interest to us, and we do not write out the corresponding equations.

Equations (1.7), (1.12) and (1.13) can be solved with respect to vector E^1 and H^1. Introducing notations

$$\underset{\sim}{n}\cdot\underset{\sim}{E}^o = E_n^o; \quad \underset{\sim}{n}\cdot\underset{\sim}{H}^o = H_n^o, \tag{1.14}$$

and expressing $\underset{\sim}{a}^1$, according to (1.3), as a function of $\underset{\sim}{a}$, we shall have:

$$\sin^2\theta\underset{\sim}{E}^1 = -NE_n^o(\underset{\sim}{n}\cos 2\theta + \underset{\sim}{a}\cos\theta) + MH_n^o[\underset{\sim}{n}\times\underset{\sim}{a}], \tag{1.15}$$

$$\sin^2\theta\underset{\sim}{H}^1 = -MH_n^o(\underset{\sim}{n}\cos 2\theta + \underset{\sim}{a}\cos\theta) - NE_n^o[\underset{\sim}{n}\times\underset{\sim}{a}]. \tag{1.16}$$

The latter formulas can be written in a somewhat different form, if we replace $\underset{\sim}{a}$ and $\underset{\sim}{a}^1$ by a vector tangent to the surface

$$\underset{\sim}{b} = \underset{\sim}{a} + \underset{\sim}{n}\cos\theta = \underset{\sim}{a}^1 - \underset{\sim}{n}\cos\theta, \tag{1.17}$$

the square of which equals

$$\underset{\sim}{b}^2 = \sin^2\theta. \tag{1.18}$$

We shall have:

$$\sin^2\theta\underset{\sim}{E}^1 = NE_n^o(\underset{\sim}{n}\sin^2\theta - \underset{\sim}{b}\cos\theta) + MH_n^o[\underset{\sim}{n}\times\underset{\sim}{b}], \tag{1.19}$$

$$\sin^2\theta\underset{\sim}{H}^1 = MH_n^o(\underset{\sim}{n}\sin^2\theta - \underset{\sim}{b}\cos\theta) - NE_n^o[\underset{\sim}{n}\times\underset{\sim}{b}]. \tag{1.20}$$

Such are the amplitude values of a wave reflected from the surface of the body as derived from the Fresnel formulas.

2. Differential geometry of the reflecting surface

Let equation of the reflecting surface in a parametric form be:

$$x = x_o(u,v); \quad y = y_o(u,v); \quad z = z_o(u,v), \tag{2.1}$$

where u,v are Gaussian coordinate parameters (curvilinear coordinates on the surface).

Assuming

$$g_{uv} = \frac{\partial x_o}{\partial u}\frac{\partial x_o}{\partial v} + \frac{\partial y_o}{\partial u}\frac{\partial y_o}{\partial v} + \frac{\partial z_o}{\partial u}\frac{\partial z_o}{\partial v} \tag{2.2}$$

and determining analogously g_{uu} and g_{vv}, we write the square of the arc element on the surface in the form of:

$$d\sigma^2 = g_{uu}du^2 + 2g_{uv}dudv + g_{vv}dv^2 \tag{2.3}$$

or shorter

$$d\sigma^2 = \sum_{u,v} g_{uv}dudv. \tag{2.4}$$

We shall utilize the notations for covariant and contra-variant components of vectors and tensors by raising and lowering the symbols with the aid of the "metric" tensor g_{uv} contained in (2.4).

If we let

$$= g_{uu}\, g_{vv} - (g_{uv})^2, \tag{2.5}$$

then the contra-variant components of the metric tensor will equal to:

$$g^{uu} = \frac{g_{vv}}{g}; \; g^{uv} = \frac{g_{uv}}{g}\; ; \; g^{vv} = \frac{g_{uu}}{g} \tag{2.6}$$

The totality of the values (2.6) is also called a tensor, which is inverse to the tensor g_{uv}. The element of the surface will be written in the form:

$$dS = \sqrt{g}\, du\, dv. \qquad (2.7)$$

In the following we shall deal with a covariant differentiation on the surface. For this, we assume

$$\frac{\partial x_o}{\partial w}\frac{\partial^2 x_o}{\partial u\,\partial v} + \frac{\partial y_o}{\partial w}\frac{\partial^2 y_o}{\partial u\,\partial v} + \frac{\partial z_o}{\partial w}\frac{\partial^2 z_o}{\partial u\,\partial v} = uv,w\ , \qquad (2.8)$$

where instead of the combination of u, v, we can also write u,u or v,v and the letter w may take on the value of u,v. The values uv,w called Christoffel's symbols, can be expressed by derivatives of g_{uv}, and namely:

$$[uv,w] = \frac{1}{2}\left(\frac{\partial g_{uw}}{\partial v} + \frac{\partial g_{vw}}{\partial u} - \frac{\partial g_{uw}}{\partial w}\right) \qquad (2.9)$$

In our case there are six Christoffel's symbols - three values of the form:

$$[uu,u] = \frac{1}{2}\frac{\partial g_{uu}}{\partial u}\ ;\ [uv,u] = \frac{1}{2}\frac{\partial g_{uu}}{\partial v}\ ,$$
$$[uu,v] = \frac{\partial g_{uv}}{\partial u} - \frac{1}{2}\frac{\partial g_{uu}}{\partial v}\ , \qquad (2.10)$$

and the other three values obtained from the preceding ones by substitution of u with v, and conversely. With their aid we form "tensorial parameters" (or "Christoffel's symbols of the second kind"), i.e. values:

$$\Gamma^p_{qr} = g^{pu}[qr,u] + g^{pv}[qr,v], \qquad (2.11)$$

where each of the letters p, q and r may take on the values u, v.

Let f(u,v) be a certain function of the point on the surface. The covariant components of its gradient on the surface will be equal to:

$$f_u = \partial f/\partial u; \quad f_v = \partial f/\partial v, \tag{2.12}$$

and the contra-variant components will be

$$f^u = g^{uu}f_u + g^{uv}f_v; \; f^v = g^{uv}f_u + g^{vv}f_v, \tag{2.13}$$

with the square of the gradient being equal to:

$$f_u f^u + f_v f^v = g^{uu}\left(\frac{\partial f}{\partial u}\right)^2 + 2g^{uv}\,\frac{\partial f}{\partial u}\,\frac{\partial f}{\partial v} + g^{vv}\left(\frac{\partial f}{\partial v}\right)^2 \tag{2.14}$$

The square of the gradient is scalar, i.e. it is not dependent upon the selection of the coordinate parameters u,v.

The second covariant derivative of the function f(u,v) differs from the usual second derivative by the terms linear in the first derivatives. We have

$$\begin{aligned} f_{uu} &= \frac{\partial^2 f}{\partial u^2} - \Gamma^u_{uu}\,\frac{\partial f}{\partial u} - \Gamma^v_{uu}\,\frac{\partial f}{\partial v}\,, \\ f_{uv} &= \frac{\partial^2 f}{\partial u \partial v} - \Gamma^u_{uv}\,\frac{\partial f}{\partial u} - \Gamma^v_{uv}\,\frac{\partial f}{\partial v}\,, \\ f_{vv} &= \frac{\partial^2 f}{\partial v^2} - \Gamma^u_{vv}\,\frac{\partial f}{\partial u} - \Gamma^v_{vv}\,\frac{\partial f}{\partial v}\,. \end{aligned} \tag{2.15}$$

It can be proved that the totality of values f_{uu}, f_{uv} and f_{vv} represents a symmetrical tensor, and the expression

$$f_{uu}\,du^2 + 2f_{uv}\,du\,dv + f_{vv}\,dv^2 \tag{2.16}$$

does not depend upon the choice of coordinates u,v.

Let us go over now to the formula for the vector components of the normal to the surface and their derivatives with respect to the parameters u and v; these latter are connected with the radii of the curvature of the normal cross-sections of the surface. We have:

$$\sqrt{g}\,n_x = \frac{\partial y_o}{\partial u}\frac{\partial z_o}{\partial v} - \frac{\partial y_o}{\partial v}\frac{\partial z_o}{\partial u} \qquad \text{etc.} \tag{2.17}$$

where letters "etc." mean two analogous expressions, obtained by a cyclic re-arrangement of letters (x, y and z).

It is obvious that

$$n_x \frac{\partial x_o}{\partial u} + n_y \frac{\partial y_o}{\partial u} + n_z \frac{\partial z_o}{\partial u} = 0, \quad n_x \frac{\partial x_o}{\partial v} + n_y \frac{\partial y_o}{\partial v} + n_z \frac{\partial z_o}{\partial v} = 0. \tag{2.18}$$

We assume

$$\begin{aligned} G_{uu} &= n_x \frac{\partial^2 x_o}{\partial u^2} + n_y \frac{\partial^2 y_o}{\partial u^2} + n_z \frac{\partial^2 z_o}{\partial u^2}, \\ G_{uv} &= n_x \frac{\partial^2 x_o}{\partial u\,\partial v} + n_y \frac{\partial^2 y_o}{\partial u\,\partial v} + n_z \frac{\partial^2 z_o}{\partial u\,\partial v}, \\ G_{vv} &= n_x \frac{\partial^2 x_o}{\partial v^2} + n_y \frac{\partial^2 y_o}{\partial v^2} + n_z \frac{\partial^2 z_o}{\partial v^2}. \end{aligned} \tag{2.19}$$

On the strength of the equations (2.18), we can replace here the usual second derivatives of x_o, y_o, z_o by the covariant ones. As a matter of fact, by assuming in (2.15) successively $f = x_o$, $f = y_o$, and $f = z_o$, multiplying by n_x, n_y and n_z, and adding, we obtain on the left-hand side the linear combination of covariant second derivatives, and on the right, the expressions (2.19), as on the right hand side, the members with the first derivatives will be cancelled as a result of (2.18). Hence, it follows that the totality of values G_{uu}, G_{uv} and G_{vv} represents a tensor, which will obviously be symmetrical.

On the strength of the same equations (2.18) taken with the opposite sign, the values G_{uv} etc. can be written in the form of:

$$- G_{uv} = \frac{\partial n_x}{\partial u}\frac{\partial x_o}{\partial v} + \frac{\partial n_y}{\partial u}\frac{\partial y_o}{\partial v} + \frac{\partial n_z}{\partial u}\frac{\partial z_o}{\partial v} . \qquad (2.20)$$

Hence it follows that

$$- \sum_{u,v} G_{uv}\, du\, dv = dn_x\, dx_o + dn_y\, dy_o + dn_z\, dz_o. \qquad (2.21)$$

Assuming that

$$dn_x = (dx_o/R) + \delta n_x \quad \text{etc.} \qquad (2.22)$$

where the infinitely small vector δn is perpendicular to the normal n and to the vector of displacement (dx_o, dy_o, dz_o), we obtain

$$- \sum_{u,v} G_{uv}\, du\, dv = d\sigma^2/R, \qquad (2.23)$$

where $d\sigma^2$ is the square of the displacement vector producible by expression (2.3). Relations (2.22) indicate that R is the radius of curvature of the intersection of surface and the plane containing the normal and displacement victor. Thus, the formula (2.23) gives us an expression for the radius of curvature, R, in dependence upon the direction of the plane of the normal cross-section.

Solving equations (2.20) with respect to derivatives of n_x, n_y and n_z with respect to u,v, we obtain:

$$\frac{\partial n_x}{\partial u} = - G_u^u \frac{\partial x_o}{\partial u} - G_u^v \frac{\partial x_o}{\partial v} ,$$
$$\frac{\partial n_x}{\partial v} = - G_v^n \frac{\partial x_o}{\partial v} - G_v^v \frac{\partial x_o}{\partial v} , \tag{2.24}$$

where values G_v^u are obtained from G_{uv} through formulas analogous to (2.13)

Designating the principal curvature radii of the normal cross-section by R_1 and R_2, we have

$$K = 1/(R_1 R_2) = G_u^u G_v^v - G_v^u G_u^v , \tag{2.25}$$

$$(1/R_1) + (1/R_2) = - G = - G_u^u - G_v^v . \tag{2.26}$$

The value K is the Gaussian curvature of the surface.

3. Cross-section of the bundle of reflected rays.

Fresnel formulas give the amplitude value of the reflected wave on the surface of a body. For finding the amplitude of the wave reflected at a certain distance from the surface, it is necessary to have the formulas for the cross-section of the bundle, passing through a surface area dS of the body, and which after reflection has traversed the given distance s. Such formulas were carried out by us [in Ref. (1)] for the case, when the incident wave is plane. In the present work we shall derive them for a general case of an arbitrary incident wave.

According to the law of reflection written in the form of (1.17), single vectors $\underset{\sim}{a}$ and $\underset{\sim}{a}^1$, characterizing the direction of the incident and reflected ray are expressed by the vector $\underset{\sim}{b}$ tangent to the surface according to the formulas:

$$a = b - n \cos \theta, \tag{3.1}$$

$$a^1 = b + n \cos \theta, \tag{3.2}$$

and moreover,

$$n \cdot b = 0; \quad b^2 = \sin^2 \theta. \tag{3.3}$$

We designate by $\omega(u,v)$ the value of phase ψ of the wave incident upon point (u,v) on the surface of the body. Since the vector $\underset{\sim}{a}$ is the gradient of the phase function ψ, the components of vector $\underset{\sim}{a}$, [which are tangent to the surface which, on the strength of (3.1), are equal to the tangents of components of vector b] can be expressed with the derivatives of ω, by u and v. These derivatives are, in turn, expressed by components of vector $\underset{\sim}{b}$. We have

$$\begin{aligned} \frac{\partial \omega}{\partial u} &= \omega_u = b_x \frac{\partial x_o}{\partial u} + b_y \frac{\partial y_o}{\partial u} + b_z \frac{\partial z_o}{\partial u} . \\ \frac{\partial \omega}{\partial v} &= \omega_v = b_x \frac{\partial x_o}{\partial v} + b_y \frac{\partial y_o}{\partial v} + b_z \frac{\partial z_o}{\partial v} . \end{aligned} \tag{3.4}$$

Combining this with the first equation (3.3), we can solve these three equations for b_x, b_y, and b_x. We obtain:

$$b_x = \omega^u \frac{\partial x_o}{\partial u} + \omega^v \frac{\partial x_o}{\partial v} \quad \text{etc.} \tag{3.5}$$

where the values ω^u, ω^v are connected with the derivatives ω_u, ω_v by relationships analogous to (2.13).

The second equation in (3.3) can be written in the form:

$$\sum_{u,v} g_{uv}\, \omega^u \omega^v = \sum_{u,v} g^{uv}\, \omega_u \omega_v = \omega_u \omega^u + \omega_v \omega^v = \sin^2 \theta. \tag{3.6}$$

Thus, the angle of incidence θ is expressed directly by ω_u, ω_v.

We examine equations

$$x = x_o + sa_x^1 \qquad \text{etc.} \tag{3.7}$$

which can be written in the form of

$$x = x_o + sb_x + s \cos\theta \, n_x \qquad \text{etc.} \tag{3.8}$$

All values on the right side, except s, represent certain known functions of point (u,v) on the surface. Considering (u,v) as constant, and varying s, we obtain the equation of the ray reflected off point u, v. The parameter s is, obviously, the path traversed by the ray after reflection. Since the phase of the incident wave at the point of reflection is $\omega(u,v)$, the phase χ of the reflected wave will then be equal to:

$$\chi = s + \omega(u,v). \tag{3.9}$$

Expressing s in (3.7) in terms of χ, we obtain:

$$\begin{aligned} x &= x_o + (\chi - \omega)a_x^1, \\ y &= y_o + (\chi - \omega)a_y^1, \\ z &= z_o + (\chi - \omega)a_z^1 \end{aligned} \tag{3.10}$$

With a constant χ formulas (3.10) represent parametric equations of the wave surface of the reflected wave.

If in formulas (3.10) we vary the values u, v within the limits (u,u + du), (v,v + dv), we shall have a certain area of the wave surface. This area can be considered as an intersection by the wave surface of the bundle of reflected rays, which is passing through an area $dS = \sqrt{g}\, du\, dv$. Since the rays are perpendicular to the wave surface, this cross-section will be normal. Designating its area with D(s)dS, we shall have

$$D(s)dS \begin{vmatrix} a_x^1 & a_y^1 & a_z^1 \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} du\ dv \qquad (3.11)$$

whence

$$D(s) = \frac{1}{\sqrt{g}} \begin{vmatrix} a_x^1 & a_y^1 & a_z^1 \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial v}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \qquad (3.12)$$

In these formulas, the values $\partial x/\partial u$ etc. denote derivatives of expressions (3.10), evaluated for a constant χ. The value of the determinant, however, will not change, if they are replaced by derivatives with constant s as was done in our work(1). Actually, we have

$$(\partial x/\partial u)_x = (\partial x/\partial u)_y - \omega_u a_x^1 \quad \text{etc.} \qquad (3.13)$$

and, as a result of such replacement, the second and third lines of the determinant will change to values proportional to the elements of the first line. Geometrically, this means that the intersection of the bundle by any surface (for example, by the surface s = const.) being projected on a plane perpendicular to the reflected ray, will produce a normal cross-section of the bundle.

4. Calculation of the determinant

A direct calculation of the determinant (3.12) involves intricate computations. Such computations may, however, be considerably simplified, if in vectors contained in the first, second and third line of the determinant, one would go from components along the axes x, y, z over to the components along two tangent directions and direction of the normal to the reflecting surface.

Suppose we have a determinant

$$\Delta = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{4.1}$$

We assume that

$$\begin{aligned} A_u &= A_x \frac{\partial x_o}{\partial u} + A_y \frac{\partial y_o}{\partial u} + A_z \frac{\partial z_o}{\partial u}, \\ A_v &= A_x \frac{\partial x_o}{\partial v} + A_y \frac{\partial y_o}{\partial v} + A_z \frac{\partial z_o}{\partial v}, \\ A_n &= A_x n_x + A_y n_y + A_z n_z, \end{aligned} \tag{4.2}$$

whence conversely

$$\begin{aligned} A_x &= A^u \frac{\partial x_o}{\partial u} + A^v \frac{\partial x_o}{\partial v} + A_n n_x, \\ A_y &= A^u \frac{\partial y_o}{\partial u} + A^v \frac{\partial y_o}{\partial v} + A_n n_y, \\ A_z &= A^u \frac{\partial z_o}{\partial u} + A^v \frac{\partial z_o}{\partial v} + A_n n_z . \end{aligned} \tag{4.3}$$

Here A^u and A^v are connected with A_u and A_v by formulas analogous to (2.13). An analogous transformation will be introduced for two other vectors, B and C, contained in the determinant.

We shall have then

$$\Delta = \frac{1}{\sqrt{g}} \begin{vmatrix} A_u & A_v & A_n \\ B_u & B_v & B_n \\ C_u & C_v & C_n \end{vmatrix}, \tag{4.4}$$

and also:

$$\Delta = \sqrt{g} \begin{vmatrix} A^u & A^v & A_n \\ B^u & B^v & B_n \\ C^u & C^v & C_n \end{vmatrix}. \tag{4.5}$$

In order to use these formulas for calculation of determinant (3.12), we must assume that

$$\begin{aligned} A_x &= a_x^1, \quad A_y = a_y^1, \quad A_z = a_z^1; \\ B_x &= \frac{\partial x}{\partial u}, \quad B_y = \frac{\partial y}{\partial u}, \quad B_z = \frac{\partial z}{\partial u}; \\ C_x &= \frac{\partial x}{\partial v}, \quad C_y = \frac{\partial y}{\partial v}, \quad C_z = \frac{\partial z}{\partial v}. \end{aligned} \tag{4.6}$$

According to (3.2) and (3.3), we obtain then:

$$A_u = \omega_u; \; A_v = \omega_v; \; A_n = \cos\theta. \tag{4.7}$$

The calculation of the new components of vectors B and C is considerably more complex. We have

$$B_x = \frac{\partial x_o}{\partial u} - \omega_n \, a_x^1 + s \frac{\partial a_x^1}{\partial u} \tag{4.8}$$

and according to (1.17)

$$B_x = \frac{\partial x_o}{\partial u} - \omega_u \, a_x^1 + s\left(\frac{\partial b_x}{\partial u} + n_x \frac{\partial(\cos\theta)}{\partial u} + \cos\theta \frac{\partial n_x}{\partial u}\right). \tag{4.9}$$

According to formula (2.20), we have

$$\frac{\partial n_x}{\partial u}\frac{\partial x_o}{\partial v} + \frac{\partial n_y}{\partial u}\frac{\partial y_o}{\partial v} + \frac{\partial n_z}{\partial u}\frac{\partial z_o}{\partial v} = - G_{uv}. \tag{4.10}$$

Moreover, this expression is symmetrical with respect to u,v.

Now we calculate the value

$$b_{uv} = \frac{\partial b_x}{\partial u}\frac{\partial x_o}{\partial v} + \frac{\partial b_y}{\partial u}\frac{\partial y_o}{\partial v} + \frac{\partial b_z}{\partial u}\frac{\partial z_o}{\partial v} . \tag{4.11}$$

As a result of formulas (3.4) this value can be written in the form:

$$b_{uv} = \frac{\partial^2 \omega}{\partial u\ \partial v} - \left(b_x \frac{\partial^2 x_o}{\partial u\ \partial v} + b_y \frac{\partial^2 y_o}{\partial u\ \partial v} + b_z \frac{\partial^2 z_o}{\partial u\ \partial v}\right), \tag{4.12}$$

whence it is evident that b_{uv} is likewise symmetrical with respect to u,v. Replacing here b_x, b_y, b_z by expressions from (3.5) and using (2.8), we can write this value in the form:

$$b_{uv} = \frac{\partial^2 \omega}{\partial u\ \partial v} - \omega^u \left[uv,\ u\right] - \omega^v \left[uv,\ v\right]. \tag{4.13}$$

Introducing according to (2.11) the "tensor parameters" Γ^p_{qr}, we can also write:

$$b_{uv} = \frac{o^2\omega}{ou\ ov} - \Gamma^u_{uv}\ \omega_u - \Gamma^v_{uv}\ \omega_v. \tag{4.14}$$

Comparing this expression with (2.15), we receive a simple result:

$$b_{uv} = \omega_{uv}, \tag{4.15}$$

where ω_{uv} is the second covariant derivative of the phase ω. This result is valid not only for notations (u,v), but also for other combinations of notations (u,u) and (v,v).

The obtained formulas enable us to find values B_u, B_v, C_u and C_v (shown further on). Into expressions for B_n and C_n there enter values

$$\beta_u = n_x \frac{\partial b_x}{\partial u} + n_y \frac{\partial b_y}{\partial u} + n_z \frac{\partial b_z}{\partial u}, \qquad (4.16)$$

$$\beta_v = n_x \frac{\partial b_x}{\partial v} + n_y \frac{\partial b_y}{\partial v} + n_z \frac{\partial b_z}{\partial v}. \qquad (4.17)$$

Let us calculate one of them. As a result of $(b \cdot n) = 0$ we have:

$$\beta_u = -\left(b_x \frac{\partial n_x}{\partial u} + b_y \frac{\partial n_y}{\partial u} + b_z \frac{\partial n_z}{\partial u} \right). \qquad (4.18)$$

In place of b_x, b_y and b_z, we substitute here expressions (3.5), and making use of (2.20), we obtain:

$$\beta_u = G_{uu}\,\omega^u + G_{uv}\,\omega^v; \qquad (4.19)$$

analogously

$$\beta_v = G_{vu}\,\omega^u + G_{vv}\,\omega^v. \qquad (4.20)$$

Now we can write out the new components of all vectors. We have

$$B_u = g_{uu} - \omega_u\,\omega_u + s(\omega_{uu} - \cos\theta\, G_{uu}),$$

$$B_v = g_{uv} - \omega_u\,\omega_v + s(\omega_{uv} - \cos\theta\, G_{uv}),$$

$$B_n = -\,\omega_u \cos\theta + s\left(G_{uu}\,\omega^u + G_{uv}\,\omega^v + \frac{\partial(\cos\theta)}{\partial u}\right); \qquad (4.21)$$

$$C_u = g_{vu} - \omega_v\,\omega_u + s(\omega_{vu} - \cos\theta\, G_{vu}),$$

$$C_v = g_{vv} - \omega_v\,\omega_v + s(\omega_{vv} - \cos\theta\, G_{vv}),$$

$$C_n = -\,\omega_v \cos\theta + s\left(G_{vu}\,\omega^u + G_{vv}\,\omega^v + \frac{\partial(\cos\theta)}{\partial v}\right). \qquad (4.22)$$

Besides, according to (4.7)

$$A_u = \omega_u; \qquad A_v = \omega_v; \qquad A_n = \cos\theta.$$

With these values of A, B, C, the determinant D(s), which gives the cross-section of the bundle of rays, will be equal to:

$$D(s) = \frac{1}{g}\begin{vmatrix} A_u & A_v & A_n \\ B_u & B_v & B_n \\ C_u & C_v & C_n \end{vmatrix}. \qquad (4.23)$$

This expression for the determinant can be considerably simplified with the aid of relations

$$\begin{aligned} A_u\,\omega^u + A_v\,\omega^v + A_n\cos\theta &= 1, \\ B_u\,\omega^u + B_v\,\omega^v + B_n\cos\theta &= 0, \\ C_u\,\omega^u + C_v\,\omega^v + C_n\cos\theta &= 0. \end{aligned} \qquad (4.24)$$

These relations can be easily checked. According to (3.6) we have

$$\omega_u\,\omega^u + \omega_v\,\omega^v = \sin^2\theta = 1 - \cos^2\theta. \qquad (4.25)$$

Taking from this expression the covariant derivative with respect to u and v (it coincides with the usual derivative), we obtain by dividing by 2

$$\begin{aligned} \omega_{uu}\,\omega^u + \omega_{uv}\,\omega^v &= -\cos\theta\,\frac{\partial(\cos\theta)}{\partial u}, \\ \omega_{vu}\,\omega^u + \omega_{vv}\,\omega^v &= -\cos\theta\,\frac{\partial(\cos\theta)}{\partial v}. \end{aligned} \qquad (4.26)$$

Substituting into (4.24) the evident expressions (4.7), (4.21) and (4.22) for the components of vectors A, B and C, and making use of (4.25) and (4.26), we are convinced in the validity of relations (4.24). The geometrical sense of these relations is obvious. They express the fact that A is a vector normal to the wave surface, while vectors B and C are perpendicular to A.

Multiplying the third column in (4.23) by cos θ, and making use of (4.24), we obtain

$$D(s)\cos\theta = \frac{1}{g}\begin{vmatrix} A_u & A_v & 1 \\ B_u & B_v & 0 \\ C_u & C_v & 0 \end{vmatrix} = \frac{1}{g}\begin{vmatrix} B_u & B_v \\ C_u & C_v \end{vmatrix} \qquad (4.27)$$

This expression acquires a more "elegant" form, if we introduce a symmetrical tensor

$$T_{uv} = g_{uv} - \omega_u\,\omega_v + s(\omega_{uv} - \cos\theta\, G_{uv}). \qquad (4.28)$$

According to (4.21) and (4.22) we shall have then

$$B_u = T_{uu}, \quad B_v = T_{uv}, \qquad (4.29)$$

$$C_u = T_{vu}, \quad C_v = T_{vv}, \qquad (4.30)$$

and the determinant (4.27) will take on the form of:

$$D(s)\cos\theta = \frac{1}{g}\begin{vmatrix} T_{uu} & T_{uv} \\ T_{vu} & T_{vv} \end{vmatrix} \qquad (4.31)$$

If we introduce the mixed components of tensor T_{uv} according to the formulas:

$$T^u_v = \sum_r g^{ur}\, T_{rv} \qquad (4.32)$$

then instead of (4.31) we can write

$$D\,(s)\cos\theta = \begin{vmatrix} T^u_u & T^u_v \\ T^v_u & T^v_v \end{vmatrix} \tag{4.33}$$

or expanding the determinant

$$D\,(s)\cos\theta = T^u_u\, T^v_v - T^u_v\, T^v_u \tag{4.34}$$

Thus, the calculation of the determinant D(s) is reduced to the calculation of the tensor T_{uv}, which presents no difficulties.

5. Differential geometry of the wave surface.

According to (3.10), equations

$$x = x_o + (\chi - \omega)a^1_x \quad \text{etc.} \tag{5.1}$$

represent, with constant χ, the parametric equations of the reflected wave surface. Every point on the wave surface corresponds to a definite point on the reflected surface, and namely, with the one that lies on one and the same ray. To these two points there correspond one and the same values of parameters u,v. Parameters u,v and phase χ can be interpreted as curvilinear coordinates in the space.

The square of the distance between two infinitely close points will be in the form of:

$$dl^2 = \sum_{u,v} g^1_{uv}\, du\, dv + d\chi^2. \tag{5.2}$$

In this expression the products of differentials $du d\chi$ and $dv d\chi$ will be absent, but the square of the differential $d\chi$ will enter with coefficient unity.

The quadratic form

$$d\tau^2 = \sum_{u,v} g^1_{uv}\, du\, dv \tag{5.3}$$

represents a square of an arc element on the wave surface.

We shall now find the coefficients of this quadratic form. Recalling equations (4.6) for vectors B and C, we can analogously to (2.8) write

$$g^1_{uu} = \underset{\sim}{B}^2; \quad g^1_{uv} = \underset{\sim}{B\cdot C}; \quad g^1_{vv} = \underset{\sim}{C}^2. \tag{5.4}$$

In calculating the scalar product and the squares of vectors $\underset{\sim}{B}$ and $\underset{\sim}{C}$, we can make use of their components (4.21) and (4.22). We shall have, for example

$$\underset{\sim}{B}^2 = \sum_{u,v} g^{uv}\, B_u\, B_v + B_n^2. \tag{5.5}$$

Using (4.24) and introducing notations:

$$\gamma^{uv} = g^{uv} + (\omega^u\, \omega^v/\cos^2\theta), \tag{5.6}$$

we may write

$$B^2 = \sum_{u,v} \gamma^{uv}\, B_u\, B_v. \tag{5.7}$$

Replacing the notations beneath the summation sign by letters p,q, and making use of (4.29), we obtain according to the rays from (5.4):

$$g^1_{uu} = \sum_{p,q} \gamma^{pq}\, T^{(1)}_{up}\, T_{uq} \tag{5.8}$$

Analogously,

$$g^1_{uv} = \sum_{p,q} \gamma^{pq} T_{up} T_{vq}, \qquad (5.9)$$

$$g^1_{vv} = \sum_{p,q} \gamma^{pq} T_{vp} T_{vq}. \qquad (5.10)$$

Thus, the coefficients of quadratic form (5.3) are expressed directly through tensor T_{uv}. By designating with g^1 the determinant

$$g^1 = g^1_{uu} g^1_{vv} - g^1_{uv} g^1_{vu} \qquad (5.11)$$

(a discriminant of a quadratic form) we have on the basis of equations (5.8) to (5.10)

$$g^1 = \text{Det}\ \gamma^{pq} (\text{Det}\ T_{uv})^2, \qquad (5.12)$$

whence

$$g^1 = g\ D(s)^2. \qquad (5.13)$$

The element dS^1 of the surface of a reflected wave, corresponding to the element dS of the reflecting surface, is equal to:

$$dS^1 = \sqrt{g^1}\ du\ dv = D(s) \sqrt{g}\ du\ dv = D(s) dS, \qquad (5.14)$$

as it should be.

Values T_{uv} are linear, and values g^1_{uv} are quadratic functions of s. With $s = 0$, we have

$$g^1_{uv}(0) = T_{uv}(0) = g_{uv} - \omega_u \omega_v. \qquad (5.15)$$

We note that this tensor is inverse to that of γ^{uv}.

With an arbitrary s, we can write

$$T_{uv}\ (s) = T_{uv}\ (0) + sT'_{uv}\ (0). \tag{5.16}$$

where, according to (4.28)

$$T'_{uv}\ (0) = \omega_{uv} - \cos\theta\ G_{uv}, \tag{5.17}$$

and also

$$g^1_{uv}\ (s) = T_{uv}\ (0) + 2s\ T'_{uv}\ (0) + s^2 \sum_{p,q} \gamma^{pq}\ T'_{up}\ (0)\ T'_{qv}\ (0) \tag{5.18}$$

We go over to the calculation of the second quadratic form, determining the curvature radii of the wave surface. The determination of it is analogous to (2.20), only instead of the vector $\underset{\sim}{n}$, we must substitute vector $\underset{\sim}{a}^1$ of the normal to the wave surface, and in place of values $\partial x_o/\partial v$ etc. -- the values $\partial x/\partial v$ etc., i.e. the components of "vector" $\underset{\sim}{c}$ (4.6). According to this determination we have

$$- G^1_{uv}\ (s) = \frac{\partial n^1_x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial a^1_y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial a^1_z}{\partial u}\frac{\partial z}{\partial v} \tag{5.19}$$

But this expression has already been found by us when calculating g^1_{uv}. Using (4.8), we can write

$$- sG^1_{uv}\ (s) = (B_x - (\partial x_o/\partial u) + \omega_u a^1_x)\ C_x + \ldots, \tag{5.20}$$

where the punctuation denotes the products of components according to axes y and z.

Hence:

$$-sG^1_{uv}\ (s) = B\cdot C - C_u = g^1_{uv}\ (s) - T_{uv}\ (s). \tag{5.21}$$

Thus, the coefficients of the first and second quadratic form are connected with tensor $T_{uv}(s)$ by a relation:

$$g^{1}_{uv}(s) + sG_{uv}(s) = T_{uv}(s). \tag{5.22}$$

From this as well as from (5.16) and (5.18) we can find also the evident expression for $G^{1}_{uv}(s)$, namely

$$- G^{1}_{uv}(s) = T'_{uv}(0) + s\sum_{pq} \gamma^{pq}\, T'_{up}(0)\, T'_{qv}(0). \tag{5.23}$$

In particular, with $s = 0$, as a result of (5.17), there will be

$$- G^{1}_{uv}(0) = \omega_{uv} - \cos\theta\, G_{uv}. \tag{5.24}$$

In this manner, for the reflected wave we have found both the first as well as the second form.

Analogous calculations can be carried out also for the incident wave. For this, it is sufficient to replace in (3.7) and in other formulas $\underset{\sim}{a}^{1}$ with an $\underset{\sim}{a}$ [formula (3.1)], and consider s as negative, so that $(-s)$ is a distance calculable along the ray up to the point incidence on the surface. We shall limit ourselves by introducing the formulas for the values of coefficients $g^{o}_{uv}(0)$ and $G^{o}_{uv}(0)$ of the first and second quadratic form of the incident wave in the point of incidence of the ray. We shall have

$$g^{o}_{uv}(0) = g_{uv} - \omega_{u}\,\omega_{v}, \tag{5.25}$$

$$- G^{o}_{uv}(0) = \omega_{uv} + \cos\theta\, G_{uv} \tag{5.26}$$

From these formulas, it is evident that values g^{o}_{uv} and

g^1_{uv} converge on the reflecting body, but those for G^o_{uv} and G^1_{uv} differ by their sign in the term containing cos θ. It is convenient to use relation (5.26) in the case, when the incident wave is plane: then $G^o_{uv} = 0$ and, consequently,

$$\omega_{uv} = - \cos \theta \, G_{uv} \tag{5.27}$$

Inserting this value in (4.28) we obtain

$$T_{uv} = g_{uv} - \omega_u \, \omega_v - 2s \cos \theta \, G_{uv} \tag{5.28}$$

Calculating the value D(s) according to formula (4.34) and using (4.25), we shall have after reduction by cos θ,

$$D(s) = \cos \theta - 2s \left(G \cos^2 \theta + \sum_{u,v} G_{uv} \, \omega^u \, \omega^v \right) + 4s^2 \cos \theta \, K. \tag{5.29}$$

Here K and G have values (2.25) and (2.26). In order to explain the geometrical sense of the sum, contained in the second term of (5.29), we note that if du and dv are components of displacement on the surface of a reflecting body in the plane of the incident ray, and d is the value of this displacement, then we have

$$\frac{du}{d\sigma} = \frac{\omega^u}{\sin \theta}; \qquad \frac{dv}{d\sigma} = \frac{\omega^v}{\sin \theta} \, . \tag{5.30}$$

Therefore, with R_o designating the radius of curvature of the intersection of the surface with the plane of incidence, we have

$$- \frac{1}{R_o} = \sum_{u,v} G_{uv} \frac{du}{d\sigma} \frac{dv}{d\sigma} = \frac{1}{\sin^2 \theta} \sum_{u,v} G_{uv} \, \omega^u \, \omega^v . \tag{5.31}$$

Inserting this value of the sum into (5.29) and expressing G and K, according to (2.25) and (2.26) through principal radii of curvature we have for the case of the incident plane wave the following expression for D(s):

$$D(s) = \cos\theta + 2s\left[\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\cos^2\theta + \frac{\sin^2\theta}{R_o}\right] + \frac{is^2}{R_1 R_2}\cos\theta. \tag{5.32}$$

This formula was derived by us in our previous work[1].

6. Reflection formula

The results obtained enable us to find (in the approximation of geometric optics) the electromagnetic field of the reflected wave. The field of the incident wave we wrote in the form

$$\underset{\sim}{E}^o e^{ik\psi}, \quad \underset{\sim}{H}^o e^{ik\psi}. \tag{6.1}$$

As ω (u,v) is the value of the phase ψ on the surface of the reflecting body, then on the surface of the body the field of the incident wave will be equal to

$$\underset{\sim}{E}^o\ (u,\ v)e^{ik\omega}; \quad \underset{\sim}{H}^o\ (u,\ v)e^{ik\omega}, \tag{6.2}$$

where E^o (u,v) and H^o (u,v) - are values of amplitudes E^o and H^o on the surface of the body. Knowing E^o (u,v) and H^o (u,v), it is possible to find from Fresnel formulas (given in Sect.1) the amplitude values of E^1 (u,v) and H^1 (u,v) of the field of the reflected wave on the surface of the body. The field of the reflected wave on this surface will be equal to

$$\underset{\sim}{E}^1\ (u,\ v)e^{ik\omega}; \quad \underset{\sim}{H}^1\ (u,\ v)e^{ik\omega}. \tag{6.3}$$

Thus, the values (6.3) can be considered as known (at least on the illuminated part of the surface, sufficiently distant from the boundary of the shadow).

We must find the field at a certain distance from the surface. For each of the components of the electromagnetic field this problem is reduced to the following: it is necessary to find function F satisfying the wave equation

$$\Delta F + k^2 F = 0 \tag{6.4}$$

and the condition of radiation, and acquiring on the surface of the body the given value

$$F = f(u, v)\, e^{ik\omega\,(u,v)}. \tag{6.5}$$

In our case k is the major parameter, and f(u,v) is a slowly variable function. The last assertion is to be understood in the sense that the derivatives, divided by k, of the function in directions tangent to the surface are small in comparison with the values of the function itself. It is easy in this case to indicate an approximate solution of our problem. Obviously, the phase of the desired function will be obtained by replacing ω with

$$\chi = \omega + s, \tag{6.6}$$

where s is the path traversed by the ray after the reflection. Its amplitude, however, will change inversely proportional to the square root of intersection area of the bundle of reflected rays. Thus, we arrive at the formula:

$$F = f(u,v)\sqrt{D(0)/D(s)}\; e^{ik\chi}, \tag{6.7}$$

where χ has the value of (6.6).

Formula (6.7) can be derived in the following manner. Let us try to find F in the form:

$$F = \sqrt{\rho e^{ik\chi'}}, \qquad (6.8)$$

where ρ and χ' - are certain functions of the coordinates, not dependent upon the parameter k. Inserting (6.8) into the wave equation (6.4), we find

$$\Delta F + k^2 F = e^{ik\chi'}\left[k^2\sqrt{\rho}\left(1 - (\text{grad }\chi')^2\right) + \frac{ik}{\sqrt{\rho}}\text{div }(\rho \text{ grad }\chi') + \Delta(\sqrt{\rho})\right] \qquad (6.9)$$

The equation of oscillations will be approximately satisfied, if in expression (6.9) terms of the k^2 and k order are equated to zero. For this the phase χ' and the amplitude square ρ must satisfy the equations

$$(\text{grad }\chi')^2 = 1, \qquad (6.10)$$

$$\text{div }(\rho \text{ grad }\chi') = 0. \qquad (6.11)$$

Let us introduce now the curvilinear coordinates u,v,χ, connected with rectangular Cartesian coordinates x, y, z by means of relations (3.10), and write equations (6.10) and (6.11) in these curvilinear coordinates. Introducing according to formulas, analogous to (2.6), the tensor g^{1uv}, inverse to that of g^{1}_{uv} determinable by formulas (5.8) to (5.10), we shall have instead of (6.10),

$$\sum_{u,v} g^{1uv}\frac{\partial\chi'}{\partial u}\frac{\partial\chi'}{\partial v} + \frac{\partial\chi'}{\partial\chi}^2 = 1 \qquad (6.12)$$

and instead of (6.11),

$$\frac{1}{\sqrt{g^1}}\left\{\sum_{u,v} \frac{\partial}{\partial v}\left(\rho \sqrt{g^1}\; g^{1uv} \frac{\partial \chi'}{\partial u} + \frac{\partial}{\partial \chi}\left(\rho \quad g^1 \frac{\partial \chi'}{\partial \chi}\right)\right)\right\} = 0. \qquad (6.13)$$

Equation (6.12) is satisfied, if we let

$$\chi' = \chi. \qquad (6.14)$$

Equation (6.13) is then reduced to the form:

$$\frac{\partial}{\partial \chi}\left(\rho \sqrt{g^1}\right) = 0, \qquad (6.15)$$

and since according to (5.13)

$$\sqrt{g^1} = \sqrt{g}D(s), \qquad (6.16)$$

where g is not dependent on χ, it will be satisfied if we assume that

$$\rho D\ (s) = \phi(u,v), \qquad (6.17)$$

where ϕ is an arbitrary function of u,v.

In order to have an agreement with (6.7), it is sufficient to assume that

$$\sqrt{\rho} = f(u,v)\sqrt{D(0)/D(s)}. \qquad (6.18)$$

Thus, we have proved that function (6.7) approximately satisfies the wave equation (6.4). Obviously, it also satisfies the radiation condition (its phase increases with a growing s). Finally, with s = 0, it is reduced to the given function (6.5). Consequently, it satisfies all the requirements that were set up.

Applying expression (6.7) to the field of the reflected wave, and adding to it the field of an incident wave, we shall obtain the reflection formula in the form:

$$E = E^{o}e^{ik\psi} + E^{1}(u, v) \quad D(0)/D (s) \, e^{ik\chi}, \tag{6.19}$$

$$H = H^{o}e^{ik\psi} + H^{1}(u, v) \quad D(0)/D (s) \, e^{ik\chi}. \tag{6.20}$$

In conclusion, we wish to mention that if the reflected body is convex, the reflection formula is applicable in the entire illuminated space sufficiently removed from the boundaries of the shadow (and at large distances from the body as well). If, however, the body is concave, then with certain values of s it is possible to transform denominator D(s) into zero (focal surfaces and lines). In the neighborhood of the focal lines and surfaces, the geometric optics and, in particular, the reflection formula, are not applicable, since the condition that an amplitude be a slowly varying function of coordinates is not fulfilled.

Conversion of reflection formula on the shadow boundary to the diffractional ones has been investigated (for the plane incident wave and for small distances from the surface of the body) in our work(1).

7. Reflection of the spherical wave from the surface of a sphere.

As an example for the application of the derived formulas, let us examine the reflection of a spherical wave from the surface of a sphere. Let r, θ, ϕ be spherical coordinates. Equation of the reflected surface is in the form of $r = a$. The part of the Gaussian parameters u,v is played by the angles θ, ϕ, so that in our general formulas we may assume

$$u = \theta; \quad v = \phi. \tag{7.1}$$

Let the source be located in point $\theta = 0$, $r = b$. The phase value of the wave incident on the surface of a sphere will then be

$$\omega(\theta,\phi) = \sqrt{a^2 + b^2 - 2ab \cos \theta}. \tag{7.2}$$

The element of the surface of the sphere is written

$$d\sigma^2 = a^2(d\theta^2 + \sin^2 \theta \, d\phi^2), \tag{7.3}$$

whence

$$g_{\theta\theta} = a^2; \quad g_{\theta\phi} = 0; \quad g_{\phi\phi} = a^2 \sin^2 \theta, \tag{7.4}$$

$$\sqrt{g} = a^2 \sin \theta, \tag{7.5}$$

$$g^{\theta\theta} = 1/a^2; \quad g^{\theta\phi} = 0; \quad g^{\phi\phi} = 1/(a^2 \sin^2 \theta). \tag{7.6}$$

According to the property of the sphere, the second differential formula will be proportional to the first, and we shall have

$$G_{\theta\theta} = - a; \quad G_{\theta\phi} = 0; \quad G_{\phi\phi} = - a \sin^2 \theta. \tag{7.7}$$

The covariant derivatives of phase ω will be equal to

$$\omega_\theta = ab \sin \theta/\omega; \quad \omega_\phi = 0, \tag{7.8}$$

and the contra-variant derivatives will be written as

$$\omega^\theta = b \sin \theta/(a\, \omega); \quad \omega^\phi = 0. \tag{7.9}$$

The incidence angle of the ray (which we shall designate now with γ, since the letter θ has already been assigned) will be determined from equations:

$$\sin\gamma = b\sin\theta/\omega; \quad \cos\gamma = (b\cos\theta - a)/\omega, \qquad (7.10)$$

resulting from (4.25). In order to calculate the second covariant derivatives of the phase, let us construct according to formulas (2.8) to (2.11) the Christoffel symbols. We have

$$\Gamma^{\theta}_{\theta\theta} = 0; \quad \Gamma^{\theta}_{\theta\phi} = 0; \quad \Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, \qquad (7.11)$$
$$\Gamma^{\phi}_{\theta\theta} = 0; \quad \Gamma^{\phi}_{\theta\phi} = \mathrm{ctg}\,\theta; \quad \Gamma^{\phi}_{\phi\phi} = 0.$$

Substituting these values into the general formulas (2.15), we obtain

$$\omega_{\theta\theta} = (ab/\omega^3)(b\cos\theta - a)(b - a\cos\theta),$$
$$\omega_{\theta\phi} = 0, \qquad (7.12)$$
$$\omega_{\phi\phi} = (ab/\omega)\sin^2\theta\cos\theta.$$

Now we can construct tensor T_{uv}. We have

$$T_{\theta\theta} = (a^2/\omega^2)(b\cos\theta-a)^2 + (sa/\omega^3)(b\cos\theta-a)(\omega^2 + b^2 - ab\cos\theta),$$
$$T_{\theta\phi} = 0, \qquad (7.13)$$
$$T_{\phi\phi} = a^2\sin^2\theta + (sa/\omega)\sin^2\theta\,(2b\cos\theta - a).$$

Let us go over now to the mixed components of T^{θ}_{θ} etc., and express $b\sin\theta$ and $b\cos\theta$ with the aid of (7.10) by a, ω and γ. We obtain

$$T^{\theta}_{\theta} = \frac{\cos\gamma}{\omega}\left((s+\omega)\cos\gamma + \frac{2s\omega}{a}\right), \qquad (7.14)$$
$$T^{\phi}_{\phi} = \frac{1}{\omega}\left(s + \omega + \frac{2s\omega}{a}\cos\gamma\right),$$

whereas $$T^{\theta}_{\phi} = T^{\phi}_{\theta} = 0.$$

According to (4.34) $D(s) \cos \gamma$ is equal to the product of values (7.14). Hence

$$\omega^2 D(s) = \left((s + \omega) \cos \gamma + \frac{2s\omega}{a}\right)\left(s + \omega + \frac{2s\omega}{a} \cos \gamma\right). \quad (7.15)$$

The expression is symmetrical with respect to s and ω.

Our results enable us at once to write out the reflection formula for the vertical component of the electric and magnetic Hertzian vector, which satisfies the scalar wave equation.

Designating with letter R the distance from the source, which equals

$$R = \sqrt{b^2 + r^2 - 2br \cos \theta} \quad (7.16)$$

we shall have for the Hertzian electric vector

$$U = \frac{e^{ikR}}{R} + N \frac{e^{ik\omega}}{\omega} \sqrt{\frac{D(0)}{D(s)}} \, e^{iks}, \quad (7.17)$$

where N is the Fresnel coefficient (1.10). For the Hertzian magnetic vector, the formula will be the same, only instead of N there will be another Fresnel's coefficient M.

Introducing for $D(s)$ the expression (7.15), and assuming for the sake of simplification that

$$2s\omega/a(s + \omega) = c_1, \quad (7.18)$$

we shall have

$$U = \frac{e^{ikR}}{R} + \frac{N}{\omega + s} \sqrt{\frac{\cos \gamma}{(\cos \gamma + c_1)(1 + c_1 \cos \gamma)}} \, e^{ik(\omega + s)}. \quad (7.19)$$

This formula can be compared with that obtained from the diffraction formulas derived in our paper(2) for the case of grazing ray incidence, and for distances from the surface of the sphere, which are small as compared with its radius. The formula indicated is reduced to the form:

$$U = \frac{e^{ikR}}{R}\left(1 + \frac{p - iq}{p - iq}\sqrt{\frac{p}{p + p_1}}\, e^{2ip_1p^2}\right) \tag{7.20}$$

Here

$$p = m \cos \gamma; \quad p_1 = mc_1; \quad q = im(\sqrt{\eta - 1}/\eta), \tag{7.21}$$

while

$$m = \sqrt[3]{ka/2}. \tag{7.22}$$

The necessary conditions of applicability of the reflection formula (7.20) are the large positive values of the magnitude p; if, however, p is of the order of unity, then the diffraction formulas will be valid.

It is not difficult to see that formula (7.20) in its due approximation coincides with (7.19). As the values c_1 and $\cos \gamma$ are small relative to unity, therefore, their product in (7.19) can be neglected. Further, the quantity $\omega + s$ in the denominator can be replaced with R. For the same quantity in the exponential function, we can use expression

$$\omega + s - R = \frac{4\omega s \cos^2 \gamma}{\omega + s + R} \tag{7.23}$$

whence approximately

$$k(\omega + s - R) = kac_1 \cos^2 \gamma = 2p_1p^2. \tag{7.24}$$

Furthermore,

$$\frac{\cos \gamma}{\cos \gamma + c_1} = \frac{p}{p + p_1} . \tag{7.25}$$

Finally, we have for small cos γ and for $\mu = 1$

$$N = (p + iq)/(p - iq). \tag{7.26}$$

If we use these approximate expressions, the agreement between (7.19) and (7.20) will be complete.

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XII. APPROXIMATE FORMULA FOR DISTANCE OF THE HORIZON IN THE PRESENCE OF SUPERREFRACTION

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A derivation is given of a formula for the distance of radiowave propagation (horizon distance) in the presence of superrefraction. The formula obtained is suitable for an atmospheric waveguide next the earth in which the modified refractive index depends on the height according to a hyperbolic law.

1. INTRODUCTION

A general formula for the attenuation factor was derived as a contour integral in our work on the theory of radiowave propagation in an inhomogeneous atmosphere.[1] The expression we obtained is applicable for the very general case of arbitrary behavior of the refractive index depending on height. The basic difficulty in using our general formula is in solving the differential equation for the height factor. This difficulty can be bypassed by using an asymptotic solution of the equation (this method is based on the presence of a large parameter in the equation). Obtaining an approximate expression for the height factor, the integrand in the contour integral can be written in explicit form and then it can be studied. A qualitative investigation of the integrand permits an estimate to be given of these distances at which the attenuation factor starts to decrease rapidly, in other words, the estimate of the horizon distance.

2. INITIAL FORMULAS

In the general case the field from a vertical and horizontal electric and magnetic dipole is expressed by means of two Hertz functions, U and W, which satisfy the same differential equations;

p. 318

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the limit conditions for U and W are also of the same type but with different values for the coefficients.[1] Each of the Hertz functions can be expressed by means of the attenuation factor V thus:

$$U = \frac{e^{iks}}{\sqrt{sa \sin \frac{s}{a}}} V, \tag{1}$$

where a is the radius of the earth; s is the horizontal distance measured along an arc of the earth's globe; $k = \frac{2\pi}{\lambda}$ is the absolute value of the wave vector.

The attenuation factor V is expressed more conveniently through the nondimensional quantities: the modified horizontal distance.

$$x = \frac{k}{2m^2} s \tag{2}$$

and the modified heights of the corresponding points (source and observation points):

$$y = \frac{k}{m} h; \; y' = \frac{k}{m} h', \tag{3}$$

where h and h' are heights in length units and m is the parameter:

$$m = \sqrt[3]{\frac{ka}{2}}. \tag{4}$$

The equivalent radius of the earth does not play the role in problems related to superrefraction that it plays in the normal refraction case; consequently, we do not introduce it here. In addition to the quantities listed, the attenuation factor V depends on the parameter q which enters into the limit conditions. The parameter q for the Hertz function U (vertical polarization) equals:

$$q = \frac{im}{\sqrt{\eta + 1}}, \tag{5}$$

where η is the complex dielectric constant of the medium. The parameter q for the Hertz function W (horizontal polarization) equals

$$q = im\sqrt{\eta - 1}\,. \tag{6}$$

In practice, we can put $q = \infty$ in the last case since both the parameters m and η are large.

Hence, the attenuation factor V is a function of the nondimensional quantities x, y, y', q:

$$V = V(x,\ y,\ y',\ q). \tag{7}$$

In addition to the attenuation factor V, it is convenient to analyze the function Ψ related thereto, whereby V is thus expressed:

$$V = 2\sqrt{\pi x}\, e^{i\frac{\pi}{4}}\, \Psi\,. \tag{8}$$

The function Ψ satisfies the differential equation

$$\frac{\partial^2 \Psi}{\partial y^2} + i\frac{\partial \Psi}{\partial x} + [y + r(y)]\ \Psi = 0\,, \tag{9}$$

where

$$r(y) = m^2(\epsilon - 1), \tag{10}$$

in which $\epsilon = \epsilon(h)$ is the air dielectric constant as a function of the height. Equation (9) is obtained by a transformation to the nondimensional quantities from the equation

$$\frac{\partial^2 \Psi}{\partial h^2} + 2ik\frac{\partial \Psi}{\partial s} + k^2\left(\frac{2h}{a} + \epsilon - 1\right)\Psi = 0, \tag{11}$$

in which the Ψ coefficient is proportional to the modified refractive index

$$M(h) = 10^6 \left(\frac{\epsilon - 1}{2} + \frac{h}{a}\right). \tag{12}$$

The Ψ coefficient in Eq. (9) is conveniently denoted by a single letter; we put

$$p(y) = y + r(y). \tag{13}$$

We will have

$$p(y) = m^2\left(\epsilon - 1 + \frac{2h}{a}\right), \tag{14}$$

so that p(y) is, in substance, the same modified refractive index but expressed through the nondimensional height y.

Using the notation of Eq. (13), Eq. (9) is written as

$$\frac{\partial^2 \Psi}{\partial y^2} + i\frac{\partial \Psi}{\partial x} + p(y)\Psi = 0\,. \tag{15}$$

The function Ψ satisfies the differential Eq. (15) and the limit condition

$$\frac{\partial \Psi}{\partial y} + q\,\Psi = 0 \text{ (for } y = 0); \tag{16}$$

at x = 0, it has a singularity of the form

$$\Psi = \frac{e^{-i\frac{\pi}{4}}}{2\sqrt{\pi x}}\left\{e^{i\frac{(y-y')^2}{4x}} + e^{i\frac{(y+y')^2}{4x}}\,\frac{y+y'+2iqx}{y+y'-2iqx}\right\}. \tag{17}$$

In the general expression[1] for the function Ψ as a contour integral the integrand was expressed through the solution of the equation

$$\frac{d^2f}{dy^2} + p(y)f = tf, \tag{18}$$

where t is a complex parameter. (These solutions were called the height factors above.)

In order to form the integrand, it is necessary to know both solutions of Eq. (18); let us denote them by $f_1(y,t)$ and $f_2(y,t)$.

These functions have the following asymptotic expressions for large y:

$$f_1(y,t) = \frac{c' e^{i\frac{\pi}{4}}}{\sqrt[4]{p(y) - t}} \exp\left[i \int_{\tau}^{y} \sqrt{p(u) - t}\, du\right], \tag{19}$$

$$f_2\,(y,t) = \frac{c'' e^{-i\frac{\pi}{4}}}{\sqrt[4]{p(y) - t}} \exp\left[-i \int_{\tau}^{y} \sqrt{p(u) - t}\, du\right]. \tag{20}$$

Here c', c'', τ are constants whose values are not essential since they drop out of the expression for Ψ. In the homogeneous atmosphere case when $p(y) = t$, the functions $f_1(y,t)$ and $f_2(y,t)$ reduce to the complex Airy functions $w_1(t - y)$ and $w_2(t - y)$, in which we can then put $c' = c'' = 1$ and $\tau = t$.

Let us put

$$D_{12}(t) = f_1 \frac{\partial f_2}{\partial y} - f_2 \frac{\partial f_1}{\partial y}. \tag{21}$$

Because of Eq. (18), which f_1 and f_2 satisfy, this quantity is independent of y.

Let us denote the values of $\frac{\partial f_1}{\partial y}$ and $\frac{\partial f_2}{\partial y}$ at $y = 0$ by $f_1'(0,t)$ and $f_2'(0,t)$ and let us form the function

$$F(t, y, y', q) = \frac{1}{D_{12}(t)} f_1(y', t) \left\{ f_2(y, t) - \frac{f'_2(0, t) + qf_2(0, t)}{f'_1(0, t) + qf_1(0, t)} f_1(y, t) \right\}. \tag{22}$$

The function Ψ determined for $y' \geqslant y$ by the contour integral[1]

$$\Psi = \frac{1}{2\pi i} \int e^{ixt} F(t, y, y', q)\, dt, \tag{23}$$

taken over the contour enclosing all the poles of the integrand in a positive direction, satisfies all the conditions set above and yields a solution to our problem.

3. NORMAL REFRACTION CASE

The normal refraction case is characterized by the modified refractive index M(h) being a monotonically increasing function of the height h and, therefore, the coefficient p(y) is a monotonically increasing function of y. In this case, $f_1(y, t)$ and $f_2(y, t)$ can be expressed approximately by the complex Airy functions of argument ξ defined by the equalities

$$\int_b^y \sqrt{p(u) - t}\, du = \frac{2}{3} (-\xi)^{\frac{3}{2}} ; \tag{24}$$

$$\int_y^b \sqrt{t - p(u)}\, du = \frac{2}{3} \xi^{\frac{3}{2}}, \tag{25}$$

where b is a root of the equation

$$p(b) = t. \tag{26}$$

The value of ξ near $y = b$ will be a holomorphic function of y, namely:

$$\xi = \sqrt[3]{p'(b)}\,(b - y) + \dots \tag{27}$$

We can put approximately

$$f_1 = \sqrt{-\frac{dy}{d\xi}}\, w_1(\xi)\,,\quad f_2 = \sqrt{-\frac{dy}{d\xi}}\, w_2(\xi) \tag{28}$$

and to the same approximation

$$\frac{\partial f_1}{\partial y} = -\sqrt{-\frac{d\xi}{dy}}\, w_1'(\xi)\,;\ \frac{\partial f_2}{dy} = -\sqrt{-\frac{d\xi}{dy}}\, w_2'(\xi)\,, \tag{29}$$

from which

$$D_{12} = -2i. \tag{30}$$

Here, replacing y by y' and ξ by ξ', we obtain expressions for $f_1(y',t)$ and $f_2(y',t)$. The value of ξ corresponding to $y = 0$ is denoted by ξ_o. Using these notations, we obtain the following approximate expression for F, defined by formula (22):

$$F = \frac{i}{2}\sqrt{-\frac{dy'}{d\xi'}}\sqrt{-\frac{dy}{d\xi}}\cdot w_1(\xi')\left\{w_2(\xi) - \frac{w_2'(\xi_o) + q\left(\frac{dy}{d\xi}\right)_o w_2(\xi_o)}{w_1'(\xi_o) + q\left(\frac{dy}{d\xi}\right)_o w_1(\xi_o)}\, w_1(\xi)\right\}. \tag{31}$$

When being substituted in formula (23), this expression can be used to calculate the field in both the shadow region and in the line-of-sight region. The attenuation factor (as well as the function Ψ) is calculated in the shadow region by a residue series corresponding to the roots of the denominator

$$w_1'(\xi_o) + q\left(\frac{dy}{d\xi}\right)_o w_1(\xi_o) = 0 . \tag{32}$$

The function Ψ is calculated directly in the line-of-sight region by using the contour integral in which the principal part of the integration will lie near real negative values of t. But the quantities ξ_o, ξ and ξ' will also be negative for negative t values. Assuming these quantities to be sufficiently large, the functions w_1 and w_2 can be replaced by their asymptotic expressions:

$$w_1(\xi) = e^{i\frac{\pi}{4}}(-\xi)^{-\frac{1}{4}} e^{i\frac{2}{3}(-\xi)^{\frac{3}{2}}}, \tag{33}$$

$$w_2(\xi) = e^{-i\frac{\pi}{4}}(-\xi)^{-\frac{1}{4}} e^{-i\frac{2}{3}(-\xi)^{\frac{3}{2}}}. \tag{34}$$

Such a substitution reduces to the use of the asymptotic expressions in Eqs. (19) and (20) for $f_1(y,t)$ and $f_2(y,t)$. Consequently, the following expression is obtained for the function F [according to formula (22)]:

$$F = \frac{i}{2} \frac{1}{\sqrt[4]{p(y) - t}\,\sqrt[4]{p(y') - t}} \left\{ \exp\left[i \int_y^{y'} \sqrt{p(u) - t}\, du\right] - \frac{q - i\sqrt{p(0) - t}}{q + i\sqrt{p(0) - t}} \exp\left[i \int_0^y \sqrt{p(u) - t}\, du + i \int_0^{y'} \sqrt{p(u) - t}\, du\right] \right\}. \tag{35}$$

This formula is a generalization of formula (6.11).[1] The latter can be obtained from Eq. (35) by substituting zero for $p(y)$.

Substitution of Eq. (35) into the contour integral yields an expression, composed of two terms, for the attentuation factor, the first of which corresponds to an incident wave and the seccnd to a wave reflected once from the earth's surface with a Fresnel coefficient. The incident wave is the superposition of a wave with the phase

$$\omega(t) = xt + \int_{y}^{y'} \sqrt{p(u) - t}\, du , \tag{36}$$

and the reflected wave is the superposition of a wave with the phase

$$\phi(t) = xt + \int_{0}^{y} \sqrt{p(u) - t}\, du + \int_{0}^{y'} \sqrt{p(u) - t}\, du . \tag{37}$$

These expressions correspond to those of geometric optics. The integrals can be evaluated by the method of stationary phase, where the phase of the incident wave will equal the extremum value of $\omega(t)$ and the phase of the reflected wave will equal the extremum value of $\phi(t)$. The function $\omega(t)$ attains its extremum value for t determined from the equation

$$\omega'(t) \equiv x - \frac{1}{2} \int_{y}^{y'} \frac{du}{\sqrt{p(u) - t}} = 0, \tag{38}$$

and the function $\phi(t)$ for t determined from

$$\phi'(t) = x - \frac{1}{2} \int_{0}^{y} \frac{du}{\sqrt{p(u) - t}} - \frac{1}{2} \int_{0}^{y'} \frac{du}{\sqrt{p(u) - t}} . \tag{39}$$

The distance of the horizon from the geometrical optics viewpoint is determined from the condition that a reflected wave with a real phase could reach up to this point. The least value of t for which this still occurs is $t = p(0)$. This value must simultaneously be a root of Eq. (39).

Therefore, the following relation must exist between x, y and y'

$$x = \frac{1}{2}\int_{0}^{y} \frac{du}{\sqrt{p(u) - p(0)}} + \frac{1}{2}\int_{0}^{y'} \frac{du}{\sqrt{p(u) - p(0)}}, \tag{40}$$

which yields the formula for distance to the horizon under normal refraction.

The more exact expression in Eq. (31) for F shows that it is already impossible to use Eq. (35) at $t = p(0)$. Actually, the quantity ξ_0 becomes zero at this value of t and it is, understandably, inadmissible to use formulas (33) and (34). Nevertheless, it can be considered that the value of x, determined from Eq. (40), approximately gives the boundary defining the line-of-sight region where the residue series is applicable. In other words, it can be considered that the field amplitude starts to decrease rapidly when x, increasing, passes through the value in Eq. (40). The terminology "horizon distance" can be used in diffraction theory in this sense.

4. ASYMPTOTIC INTEGRATION OF A DIFFERENTIAL EQUATION WITH A COEFFICIENT HAVING A MINIMUM

The modified index of refraction will not be a monotonic function of the height in the presence of superrefraction but it will have one or more minimums corresponding to the separate waveguide channels. We will consider the case of a single minimum; we will call the corresponding height the inversion height and will denote it by h_i.

The coefficient $p(y)$ proportional to $M(h)$ of the differential equation

$$\frac{d^2f}{dy^2} + p(y)f = tf \tag{41}$$

will also have one minimum at $y = y_i$ corresponding to $h = h_i$.

We will consider $p(y)$ to be an analytic function of y. The equation $p(y) = t$ will have two roots in the region interesting us: $y = b_1$ and $y = b_2$.

Both roots will be real for t real and lying between $p(0)$ and $p(y_i)$; the roots can be complex for other t values.

We must have such an asymptotic expression for the functions $f_1(y)$, $f_2(y)$ as would be valid uniformly for all the values of y and t considered, with the exception of the value $t = p(y_i)$ at which the roots b_1 and b_2 coincide.

The expressions used in section 2 for f_1 and f_2 in terms of the Airy function are not applicable here. Its validity was based on Eq. (41) reducing approximately to

$$\frac{d^2w}{d\xi^2} - \xi w = 0, \tag{42}$$

in which the coefficient for the unknown function now has the same monotonic character as in the initial equation, by means of the substitution Eqs. (24)-(25) which defines ξ as a holomorphic function of y. Now, we must take as the standard equation

$$\frac{d^2g}{d\zeta^2} + (\frac{1}{4}\zeta^2 + \nu)\, g = 0 \tag{43}$$

for the parabolic cylinder function instead of Eq. (42) for the Airy function, since this is the most simple equation in which the coefficient for the unknown function has the same charactoer (with a single minimum) as does the coefficient $p(y)$. It is necessary to select the substitution relating ζ to y so that the quantity $p(y) - t$ becomes zero simultaneously with the quantity $\frac{1}{4}\zeta^2 + \nu$ and so that the correct asymptotic expressions would be obtained for large values of these quantities. The substitution

$$\int_{b_1}^{y} \sqrt{p(y) - t}\, dy = \frac{1}{2} \int_{-2i\sqrt{\nu}}^{\zeta} \sqrt{\zeta^2 + 4\nu}\, d\zeta, \tag{44}$$

satisfies these conditions under the condition that the parameter ν is chosen so that

$$\int_{b_1}^{b_2} \sqrt{p(y) - t}\, dy = \frac{1}{2} \int_{-2i\sqrt{\nu}}^{2i\sqrt{\nu}} \sqrt{\zeta^2 + 4\nu}\, d\zeta. \tag{45}$$

The integral in the right side of (45) equals

$$\frac{1}{2} \int_{-2i\sqrt{\nu}}^{2i\sqrt{\nu}} \sqrt{\zeta^2 + 4\nu}\, d\zeta = i\pi\nu. \tag{46}$$

Consequently, Eq. (45) can be written thus

$$i\pi\nu = \int_{b_1}^{b_2} \sqrt{p(y) - t}\, dy. \tag{47}$$

It gives ν as a function of t. This function will be holomorphic near $t = p(y_i)$, namely, we will have

$$\nu = \frac{p(y_i) - t}{\sqrt{2p''(y_i)}} + \ldots \tag{48}$$

Putting

$$S = \int_0^y \sqrt{p(y) - t}\, dy \tag{49}$$

$$S_o = \frac{1}{2}\int_0^{b_1} \sqrt{p(y) - t}\, dy + \frac{1}{2}\int_0^{b_2} \sqrt{p(y) - t}\, dy, \tag{50}$$

we can write the substitution (44) as

$$S - S_o = \frac{1}{2}\int_0^{\zeta} \sqrt{\zeta^2 + 4\nu}\, d\zeta. \tag{51}$$

The first part of this expression equals

$$\frac{1}{2}\int_0^{\zeta} \sqrt{\zeta^2 + 4\nu}\, d\zeta = \frac{1}{4}\zeta\sqrt{\zeta^2 + 4\nu} + \ln(\zeta + \sqrt{\zeta^2 + 4\nu}) - \frac{\nu}{2}\ln 4\nu. \tag{52}$$

Hence we can conclude that the quantity $S - S_0 + \frac{\nu}{2}\ln\nu$ will be a holomorphic function of ν near $\nu = 0$ for $\zeta > 0$ and the quantities $S - S_o - \frac{\nu}{2}\ln\nu$ and S will be holomorphic for $\zeta < 0$. But since we know $\zeta < 0$ at $y = 0$ (on the earth's surface), the sum $S_o + \frac{\nu}{2}\ln\nu$ will be a holomorphic function of ν. This remark will be needed later.

The solutions of Eqs. (41) and (43), in the asymptotic approximations under consideration, are related by the relation

$$f = \sqrt{\frac{dy}{d\zeta}}\, g\ . \tag{53}$$

Solutions of Eq. (43) are functions which are expressed by means of the parabolic cylinder function $D_n(z)$ which satisfies the equation

$$\frac{d^2 D_n(z)}{dz^2} + (n + \frac{1}{2} - \frac{1}{4} z^2)\, D_n(z) = 0\,. \tag{54}$$

The functions $D_n(z)$ have been well investigated. We will not enumerate their properties but will refer to the book by Whittaker and Watson "Course of Mordern Analysis" where the principal formulas are given. The following series can be taken as adefinition $D_n(z)$

$$D_n(z) = \frac{2^{-\frac{n}{2}-1}}{\Gamma(-n)} e^{-\frac{z^2}{4}} \sum_{m=0}^{\infty} \frac{\Gamma(\frac{m-n}{2})}{\Gamma(m+1)} \cdot 2^{\frac{m}{2}} (-z)^m\,. \tag{55}$$

Equation (43) is obtained from Eq. (54) by replacing z by ζ $\exp(-i\frac{\pi}{4})$ and $n + \frac{1}{2}$ by $i\nu$. The functions

$$g_1(\zeta) = D_{i\nu - \frac{1}{2}}(e^{-i\frac{\pi}{4}}\zeta)\,, \tag{56}$$

$$g_2(\zeta) = D_{-i\nu - \frac{1}{2}}(e^{i\frac{\pi}{4}}\zeta)\,. \tag{57}$$

will be solutions of Eq. (43). The quantities $g_1(\zeta)$ and $g_2(\zeta)$ will be complex conjugates for real ν and ζ.

There results from the properties of $D_n(z)$

$$g_1(-\zeta) = e^{-\nu\pi - i\frac{\pi}{2}} g_1(\zeta) + \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2} - i\nu)} e^{-\frac{\nu\pi}{4} + i\frac{\pi}{4}} g_2(\zeta) \quad (58)$$

$$g_2(-\zeta) = e^{-\nu\pi + i\frac{\pi}{2}} g_2(\zeta) + \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2} + i\nu)} e^{-\frac{\nu\pi}{4} - i\frac{\pi}{4}} g_1(\zeta). \quad (59)$$

Asymptotic expressions for $g_1(\zeta)$ and $g_2(\zeta)$ are essential to us. In the region adjoining the positive real axis, we have

$$g_1(\zeta) = e^{\frac{\pi\nu}{4} + i\frac{\pi}{8}} e^{i\frac{\zeta^2}{4}} \zeta^{i\nu - \frac{1}{2}} \left(1 + \frac{i\nu^2 - 2\nu - i\frac{3}{4}}{2\zeta^2} + \right) \ldots \quad . \quad (60)$$

Using Eq. (52), we can also write

$$g_1(\zeta) = e^{\frac{\pi\nu}{4} + i\frac{\pi}{8}} e^{-i\frac{\nu}{2} + i\frac{\nu}{2}\ln\nu} \frac{1}{\sqrt[4]{\zeta^2 + 4\nu}} \exp\left[\frac{i}{2}\int_0^{\zeta} \sqrt{\zeta^2 + 4\nu}\, d\zeta\right]. \quad (61)$$

The latter expression is valid also for large ν. The asymptotic expression for $g_2(\zeta)$ is obtained by replacing i by $-i$.

In order to obtain a formula valid near the negative real axis, we must use relation (58). We will have

$$g_1(\zeta) = e^{-\frac{3\nu\pi}{4} - i\frac{3\pi}{8}} e^{-i\frac{\nu}{2} + \frac{i\nu}{2}\ln\nu} \frac{1}{\sqrt[4]{\zeta^2 + 4\nu}} \exp\left[\frac{i}{2}\int_0^{\zeta} \sqrt{\zeta^2 + 4\nu}\, d\nu\right] + \quad (62)$$

$$+\frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2}-i\nu)} e^{-\frac{\nu\pi}{4}+i\frac{\pi}{8}} e^{i\frac{\nu}{2}-\frac{i\nu}{4}\ln\nu} \frac{1}{\sqrt[4]{\zeta^2+4\nu}} \exp\left[\frac{i}{2}\int_0^{\zeta}\sqrt{\zeta^2+4\nu}\,d\nu\right].$$

We are now in a position to construct the solution of Eq. (41) which satisfies all the requirements.

Let us put

$$c_1(\nu) = e^{i\frac{\pi}{8}-\frac{\pi\nu}{4}} e^{i(\frac{\nu}{2}-\frac{\nu}{2}\ln\nu - S_o)} . \tag{63}$$

Because of the properties of S noted above, the exponential in Eq. (63) is a holomorphic function of ν also near $\nu = 0$.

The function

$$f_1(y,t) = c_1(\nu)\sqrt{2\frac{dy}{d\zeta}}\, g_1(\zeta). \tag{64}$$

will be a suitable solution of the equation for the height factor. Above the inversion layer (for $S - S_o \gg 1$) this function has the asymptotic expression

$$f_1(y,t) = \frac{e^{i\frac{\pi}{4}}}{\sqrt[4]{p(y)-t}} e^{iS - 2iS_o}, \tag{65}$$

which results for Eq. (61).

Below the inversion layer (for $S_o - S \gg 1$) the asymptotic expression for $f_1(y,t)$ will be

$$f_1(y,t) = \chi_1(\nu)\frac{e^{i\frac{\pi}{4}}}{\sqrt[4]{p(y)-t}} e^{iS - 2iS_o} + e^{-\nu\pi}\frac{e^{-i\frac{\pi}{4}}}{\sqrt[4]{p(y)-t}} e^{-iS}, \tag{66}$$

where we put

$$\chi_1(\nu) = \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2} - i\nu)} e^{-\frac{\nu\pi}{2}} e^{i(\nu - \nu \ln \nu)} . \tag{67}$$

Using the known asymptotic expression for the function $\Gamma(\frac{1}{2} - i\nu)$, it is easy to show that the function $\chi_1(\nu)$ tends to unity for large positive values of ν. Inasmuch as the second term in Eq. (66) becomes small in comparison with the first for $\nu \gg 1$, both expressions for $f_1(y,t)$ will then agree in form. However, it is essential that our expressions for $f_1(y,t)$ be valid not only for large, but also for small, values of ν down to $\nu = 0$ and that they be holomorphic functions of ν near $\nu = 0$.

The appropriate expressions for $f_2(y,t)$ are obtained from the preceding by substituting $-i$ for i. In order to write them explicitly, let us put

$$c_2(\nu) = e^{-i\frac{\pi}{8} - \frac{\pi\nu}{4}} e^{-i(\frac{\nu}{2} - \frac{\nu}{2} \ln \nu - S_o)} , \tag{68}$$

$$\chi_2(\nu) = \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2} + i\nu)} e^{-\frac{\nu\pi}{2}} e^{-i(\nu - \nu \ln \nu)} . \tag{69}$$

Then

$$f_2(y,t) = c_2(\nu)\sqrt{2\frac{dy}{d\zeta}}\, g_2(\zeta) , \tag{70}$$

and the asymptotic expressions for $f_2(y,t)$ will be following

$$f_2(y,t) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt[4]{p(y) - t}} e^{-iS + 2iS_o}; \text{ for } S - S_o \gg 1 \tag{71}$$

$$f_2(y,t) = \chi_2(\nu) \frac{e^{-i\frac{\pi}{4}}}{\sqrt[4]{p(y) - t}} e^{-iS + 2iS_o} + e^{-\nu\pi} \frac{e^{i\frac{\pi}{4}}}{\sqrt[4]{p(y) - t}} e^{iS} \tag{72}$$

$$\text{for } S_o - S \gg 1$$

Hence, the problem of the asymptotic integration of the height factor equation has been solved.

5. INVESTIGATION OF THE ATTENUATION FACTOR

We must now substitute the expressions found for $f_1(y,t)$ and $f_2(y,t)$ into formula (22) for F and we must investigate the attenuation factor V or the function Ψ related thereto. For simplicity of writing, we will limit ourselves to the $q = \infty$ case, which corresponds to horizontal polarization. The function F becomes in this case

$$F(t, y, y', \infty) = \frac{1}{D_{12}(t)} f_1(y', t) \left\{ f_2(y,t) - \frac{f_2(0,t)}{f_1(0,t)} f_1(y,t) \right\}. \tag{73}$$

The Wronskiian D_{12} for the functions (64) and (70) equals the constant value

$$D_{12} = -2i, \tag{74}$$

which is most easily derived from the asymptotic expressions (65) and (61). We will assume that $y' > y_i$ so that $S(y') - S_o \gg 1$ and let us consider two cases: when the second height is also high and when it is below the inversion layer. In the first case, we will consider

$S(y) - S_o \gg 1$, which permits expressions (65) and (70) for f_1 and f_2 to be used. In the second case, we shall consider $S_o - S(y) \gg 1$ and we shall use expressions (66) and (72).

In the first case, we shall have

$$F = \frac{i}{2} \frac{e^{iS(y') - 2iS_o}}{\sqrt[4]{p(y') - t}\,\sqrt[4]{p(y) - t}} \left\{ e^{-iS + 2iS_o} + \frac{e^{-\pi\nu} - i\chi_2 e^{2iS_o}}{e^{-\pi\nu} + i\chi_1 e^{-2iS_o}} e^{iS - 2iS_o} \right\} \tag{75}$$

The separate terms of this expression admit of an interpretation on the basis of geometric optics. It is evident that a wave going from above downward must have the phase factor e^{-iS} and a wave going from below upward must have the phase factor e^{iS}. Expression (75) shows that there is only one wave going from above downward, namely, an incident wave with the total phase

$$\omega(t) = xt + S(y') - S(y) \tag{76}$$

we added the term xt here from the exponential in integral (23). This phase agrees with the phase Eq. (36) of the normal refraction case, as is natural, since this wave did not reach the inversion layer.

As regards the waves going upward from below, they will be an innumberable set; these waves are obtained by expanding the second term of Eq. (75) in a power series in $e^{-\pi}$. They will correspond to waves, multiply reflected from the earth's surface and from the inversion layer. The phase of waves reflected once from the earth's surface will be

$$\phi(t) = xt + S(y') + S(y) + \mathrm{arc}\frac{\chi_2}{\chi_1}. \tag{77}$$

This expression differs from Eq. (37) in its last term which cannot be obtained from geometric optics. This term equals

$$\operatorname{arc}\frac{\chi_2}{\chi_1} = \operatorname{arc}\frac{\Gamma(\frac{1}{2} - i\nu)}{\Gamma(\frac{1}{2} + i\nu)} + 2\nu \ln \nu - 2\nu. \tag{78}$$

It becomes zero for large positive ν but it plays an important part for small ν since, because of it, the whole phase $\phi(t)$ remains a holomorphic function of ν near $\nu = 0$, in other words, near $t = p(y_i)$.

Now let us analyze the case when the point y is below the inversion layer, where $S_o - S \gg 1$.

Using expressions (66) and (72) and the equality

$$\chi_1(\nu)\,\chi_2(\nu) - e^{-2\pi\nu} = 1, \tag{79}$$

we obtain after certain computations

$$F = \frac{e^{iS(y') - 2iS_o}}{\sqrt[4]{p(y') - t}\,\sqrt[4]{p(y) - t}}\;\frac{\sin S(y)}{\chi_1 e^{-2iS_o} - ie^{-\pi\nu}}. \tag{80}$$

In this case, there is not one but an innumerable quantity of waves going downward from above since waves reflected from the inversion layer as well as from the upper boundaries are added to the incident wave. Moreover, there is an infinite quantity of waves reflected from the earth and going upward from below. All these waves are obtained formally by expanding Eq. (80) in a geometric progression in powers of $e^{-\pi}$.

The total phase of waves not reflected from the earth equals

$$\omega(t) = xt + S(y') - S(y) - \operatorname{arc}\chi_1 \tag{81}$$

or

$$\omega(t) = xt + S(y') - S(y) + \text{arc}\frac{\chi_2}{\chi_1}, \tag{82}$$

and the total phase of a wave reflected once is

$$\phi(t) = xt + S(y') + S(y) + \frac{1}{2}\text{arc}\frac{\chi_2}{\chi_1}. \tag{83}$$

The expression for $\omega(t)$ does not agree with Eqs. (36) or (76), which is natural, since the incident wave passed through the inversion layer. Expression (83) differs from Eq. (77) by the additional term having a factor $\frac{1}{2}$.

Up to now, we spoke of the phases of the different terms of the integrand. An integral over t in the attenuation factor corresponds to each such term. If these integrals are evaluated by the method of stationary phase, then each one gives a term in the attenuation factor which represents a wave with a phase equal to the extremum value of the phase of the integrand.

It is understood that we use such a method of evaluating the attenuation factor only in the line-of-sight region; residue series must be used in the shadow region.

6. FORMULA FOR THE DISTANCE

We defined the horizon distance for normal refraction (section 2) as such a value of the horizontal range x as would give the boundary between the region of applicability of the reflection formula and the region of applicability of the residue series. For this value of x, the extremum of the phase of the reflected wave must be the least value of t for which the phase itself is still real.

There are many reflected waves in the presence of superrefraction. But we can expect that the principal part will be played by a wave reflected once form the earth's surface. Inasmuch as the "horizon distance" is not a strictly defined concept, we rightly make it more precise by interpreting it as the horizon distance for a single reflected wave.

The phases of a single reflected wave are found in sec. 4. According to Eqs (77) and (83), we will have for $y' > y_i$, $y > y_i$

$$\phi(t) = xt + \int_0^{y'} \sqrt{p(u) - t}\, du + \int_0^{y} \sqrt{p(u) - t}\, du + \operatorname{arc} \frac{\chi_2}{\chi_1} \tag{84}$$

and for $y' > y_i$, $y < y_i$

$$\phi(t) = xt + \int_0^{y'} \sqrt{p(u) - t}\, du + \int_0^{y} \sqrt{p(u) - t}\, du + \frac{1}{2} \operatorname{arc} \frac{\chi_2}{\chi_1}. \tag{85}$$

These formulas can be combined by putting

$$S^*(y,t) = \int_0^{y} \sqrt{p(u) - t}\, du + \frac{1}{2} \operatorname{arc} \frac{\chi_2}{\chi_1} \quad (y > y_i). \tag{86}$$

$$S^*(y,t) = \int_0^{y} \sqrt{p(u) - t}\, du \quad (y < y_i). \tag{87}$$

Then, both for $y > y_i$ and for $y < y_i$ we will have

$$\phi(t) = xt + S^*(y't) + S^*(y,t). \tag{88}$$

Let us note that S^* is a holomorphic function of t near $t = p(y_i)$.

Reasoning as in sec. 2, we obtain the following expression for the horizon distance

$$x = -\left\{\frac{\partial S^*(y',t)}{\partial t} + \frac{\partial S^*(y,t)}{\partial t}\right\}_{t=p(y_i)}. \tag{89}$$

Let us write this expression in a more explicit form. Acording to Eq. (48) near $t = p(y_i)$ we will have

$$\nu = \frac{p(y_i) - t}{\sqrt{2p''(y_i)}} . \tag{90}$$

On the other hand, near $\nu = 0$

$$\frac{1}{2} \arc \frac{\Gamma(\frac{1}{2} - i\nu)}{\Gamma(\frac{1}{2} + i\nu)} = (C + 2 \ln 2)\nu + \ldots \tag{91}$$

and therefore

$$\frac{1}{2} \arc \frac{\chi_2}{\chi_1} = \nu \, (C - 1 + \ln 4\nu) + \ldots , \tag{92}$$

where $C = 0.577$ is the Euler constant. Consequently, for $y > y_i$

$$-\frac{\partial S^*}{\partial t} = \frac{1}{2} \int_0^y \frac{du}{\sqrt{p(u) - t}} + \frac{1}{\sqrt{2p''(y_i)}} (C + \ln 4\nu) . \tag{93}$$

This expression has a limit for $t \to p(y_i)$, $\nu \to 0$. The last term is absent for $y < y_i$ and the value $t = p(y_i)$ can be substituted directly into the integral. Consequently, for $y < y_i$, we will have

$$-\frac{\partial S^*}{\partial t} = \frac{1}{2} \int_0^y \frac{du}{\sqrt{p(u) - p(y_i)}} . \tag{94}$$

The presence of the second term in formula (85) specifies the dependence of the horizon distance on the wavelength. In order to clarify this dependence, let us turn form the modified x, y coordinates to the usual s, h coordinates, where s is the horizontal range and h is the height.

Denoting the modified refractive index without the 10^6 by $\mu(h)$, we will have

$$p(y) = 2m^2\mu(h), \tag{95}$$

where m is the quantity Eq. (4). We introduce the parameter τ instead of t by means of the relation

$$t = 2m^2\tau . \tag{96}$$

Then

$$\int_0^y \sqrt{p(u) - t}\, du = k \int_0^h \sqrt{2\mu(h) - 2\tau}\, dh, \tag{97}$$

$$xt = ks\tau . \tag{98}$$

Now, the quantity ν will equal approximately

$$\nu = \frac{k}{\sqrt{\mu''(h_i)}} [\mu(h_i) - \tau] . \tag{99}$$

The distance formula is obtained from the condition

$$\frac{1}{k}\frac{\partial\phi}{\partial\tau} = 0 \text{ for } \tau = \mu(h_i) , \tag{100}$$

where the phase ϕ is assumed to be expressed by the new quantities.

Let us put

$$F(h) = \int_0^h \frac{dh}{\sqrt{2\mu(h) - 2\mu(h_i)}} \text{ (for } h < h_i). \tag{101}$$

$$F(h) = \lim_{\tau \to \mu(h_i)} \left\{ \int_0^h \frac{dh}{\sqrt{2\mu(h) - 2\tau}} + \frac{1}{\sqrt{\mu''(h_i)}} \left[C + \ln \frac{4k(\mu(h_i) - \tau)}{\sqrt{\mu''(h_i)}} \right] \right\} \quad (102)$$

(for $h > h_i$)

Then the formula for the horizon distance obtained from condition Eq. (100) is written as

$$s = F(h') + F(h) . \quad (103)$$

Let us compare the values of the horizon distance for identical heights but for different wave lengths. The wavelength enters into the expression for $F(h)$ only for $h > h_i$ and only into the logarithmic term. Let the horizon distance equal s_1 for $\lambda = \lambda_1 = \frac{2\pi}{k_1}$ and s_2 for $\lambda = \lambda_2 = \frac{2\pi}{k_2}$. Comparing the difference of expressions (103), we obtain

$$s_2 - s_1 = \frac{2}{\sqrt{\mu''(h_i)}} \ln \frac{k_2}{k_1} = \frac{2}{\sqrt{\mu''(h_i)}} \ln \frac{\lambda_1}{\lambda_2} \text{ for } h > h_i \quad (104)$$

$$s_2 - s_1 = \frac{1}{\sqrt{\mu''(h_i)}} \ln \frac{k_2}{k_1} = \frac{1}{\sqrt{\mu''(h_i)}} \ln \frac{\lambda_1}{\lambda_2} \text{ for } h > h_i \quad (105)$$

This difference depends only on the behavior of the modified refractive index near its minimum except for the ratio of the wave lengths.

Let us apply our general formula to the case when the modified refractive index $\mu(h)$ depends on the height according to a hyperbolic law

$$\mu(h) = \mu(h_i) + \frac{1}{a} \frac{(h - h_i)^2}{h + \tau} , \quad (106)$$

where a is the radius of the earth's globe; l is a parameter. In this case

$$\mu''(h_i) = \frac{2}{a(h_i + l)} \, . \tag{107}$$

The integrals in $\phi(t)$ will be elliptic but they are evaluated elementarily for $\tau = \mu(h_i)$ and we obtain the following expressions for **F(h)**:

$$F(h) = -\sqrt{2a(h+l)} + \sqrt{2al} + \tag{108}$$

$$+ \sqrt{\frac{a(h_i + l)}{2}} \left\{ \ln \frac{\sqrt{h_i + l} + \sqrt{h + l}}{\sqrt{h_i + l} - \sqrt{h + l}} - \ln \frac{\sqrt{h_i + l} + \sqrt{l}}{\sqrt{h_i + l} - \sqrt{l}} \right\}$$

for $h < h_i$

$$F(h) = \sqrt{2a(h+l)} + 2al - \tag{109}$$

$$- \sqrt{\frac{a(h_i + l)}{2}} \left\{ \ln \frac{\sqrt{h + l} + \sqrt{h_i + l}}{\sqrt{h + l} - \sqrt{h_i + l}} + \ln \frac{\sqrt{h_i + l}\sqrt{+l}}{\sqrt{h_i + l}\sqrt{-l}} \right\} + \Delta s,$$

for $h > h_i$

where

$$\Delta s = \sqrt{\frac{a(h_i + l)}{2}} \left\{ C_1 + \frac{1}{2} \ln \frac{2k^2 (h_i + l)^3}{a} \right\} . \tag{110}$$

Here

$$C_1 = 7 \ln 2 - 4 + C = 1.429 \, . \tag{111}$$

For comparison, let us note that the horizon distance in the absence of refraction equals, as is known,

$$s' = \sqrt{2ah'} + \sqrt{2ah} . \qquad (112)$$

Hence, the increase in the horizon distance because of refraction equals

$$s - s' = \left[F(h') - \sqrt{2ah'}\right] + \left[F(h) - \sqrt{2ah}\right] . \qquad (113)$$

We assumed in all the preceding reasoning that the heights h and h' are small in comparison with the radius of the earth a. But the preceding formulas are applicable when a wave comes from infinity (for example, from the sun). The difference $F(h') - \sqrt{2ah'}$ has a finite limit for $h' \to \infty$, namely:

$$\lim_{h' \to \infty} \left[F(h') - \sqrt{2ah'}\right] = \qquad (114)$$
$$= \sqrt{2a\iota} - \sqrt{\frac{a(h_i + \iota)}{2}} \ln \frac{\sqrt{h_i + \iota} + \sqrt{\iota}}{\sqrt{h_i + \iota} - \sqrt{\iota}} + \Delta s .$$

Replacing the first two terms in Eq. (112) by their limit values, we obtain the following expressions for the increase in the horizon distance:

$$s - s' = 2\sqrt{2a\iota} - \sqrt{2a(h + \iota)} - \sqrt{2ah} + \Delta s + \qquad (115)$$

$$+ \sqrt{\frac{a(h_i + \iota)}{2}} \ln \frac{\sqrt{h_i + \iota} + \sqrt{h + \iota}}{\sqrt{h_i + \iota} - \sqrt{h + \iota}} - 2 \ln \frac{\sqrt{h_i + \iota} + \sqrt{\iota}}{\sqrt{h_i + \iota} - \sqrt{\iota}}$$

for $h < h_i$

$$s - s' = 2\sqrt{2a\iota} + \sqrt{2a(h+\iota)} - \sqrt{2ah} + 2\Delta s - \tag{116}$$

$$-\sqrt{\frac{a(h_i+\iota)}{2}}\left\{\ln\frac{\sqrt{h+\iota}+\sqrt{h_i+\iota}}{\sqrt{h+\iota}-\sqrt{h_i+\iota}} + 2\ln\frac{\sqrt{h_i+\iota}+\sqrt{\iota}}{\sqrt{h_i+\iota}-\sqrt{\iota}}\right\}$$

The "lead angle"

$$\delta = \frac{s-s'}{a}. \tag{117}$$

corresponds to this increase in the distance. Since the present theory does not take refraction in the high layers of the atmosphere into account, it is necessary to add the value of normal refraction on the horizon to Eq. (117) for a comparison with the observed lead angle.

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XIII. ON RADIOWAVE PROPAGATION NEAR THE HORIZON WITH SUPERREFRACTION

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This work is devoted to the computation of anomalous radiowave propagation near the horizon when an inversion layer exists near the earth (which is invariant in the horisontal directions) for several typical examples. Curves are constructed for the attenuation factor in the case when the transmitting antenna is situated high above the inversion layer and the receiving antenna is within the inversion layer at a low elevation (or conversely).

The results obtained indicate the expediency of introducing the horizon in analyzing very remote propagation, they give an estimate of the possible values of the attenuation factor at the horizon and also indicate the dependence of the attenuation factor near the horizon on the distance and wavelength. The results obtained can be of value in analyzing the propagation of decimeter, centimeter and shorter wavelengths in the troposphere.

1. INTRODUCTION

The theory of radiowave propagation above a spherical earth in the presence of an inhomogeneous atmosphere for which the refractive index depends only on the height was worked out in the work of V.A. Fock (1,2). An investigation was given in the second of these works, of the attenuation factor in an inhomogeneous atmosphere near the horizon, where the concept of the horizon is defined for an inhomogeneous atmosphere of any kind. The definition of the horizon introduced in (2) in the case of an inhomogeneous atmosphere without an inversion of the reduced index of refraction coincides with the boundaries of the shadow which results from the laws of geometric optics. If an inversion of the reduced index of refraction exists, then the horizon is found from more exact wave considerations; in this case, its statement depends on the wavelength.

If it is assumed that the attenuation factor decreases rapidly with distance beyond the horizon, then (as was done in (2)) the range of the horizon can conditionally be considered to determine the range of radiowave propagation. Therefore, a simple formula is obtained for the range of radiowave propagation with super-refraction. The heights of the receiving and transmitting antennas, the wavelength and the parameters characterizing the M-profile all enter into this formula. The range formula for a reduced index of refraction dependent on the height according to a hyperbolic law ((2), § 5) assumes an especially simple form.

The analysis of very long propagation using the horizon concept, given in (2), requires certain improvements, however. First of all, it is desirable to clarify which values the attenuation factor at the horizon assumes and how the attenuation factor near the horizon depends on the distance, the wavelength and the parameters of the inversion layer (the height of this layer, its average gradient, etc.). To do this, it is evidently necessary to evaluate the attenuation factor in certain particular cases inasmuch as this problem is not subject to solution in a general form. Hence, if we explain how rapidly the attenuation factor decreases in the shadow region (beyond the horizon) and how rapidly it increases to a value of the order of unity when departing from the horizon into the line-of-sight region, then we thereby confirm to what degree the horison determines the range of radiowave propagation in practical cases.

In view of the enormous tedium involved in computation of the attenuation factor during super-refraction, the calculations can only be made for a small number of typical cases. Here it is impossible to perform any exhaustive calculations, as for normal radiowave propagation. Hence, we were limited to the calculation of the attenuation factor as a function of the nondimensional coordinate ζ in four cases which enabled the dependence of the attenuation factor on the horizontal distance between points, for a fixed M-curve and for fixed heights of corresponding points to be constructed for four wavelengths, referred as 1:3:9:27 (see Section 7).

In this way, it appears to be possible to make more precise the meaning of the range of the horizon and the range of propagation and to answer a number of questions formulated above, in particular, the question of the dependence of the very-long propagation phenomenon on the wavelength.

Let us recall that the analysis of anomalous propagation given in (2) is applicable if and only if one of the corresponding points is above the inversion layer near the earth while the other point can be either within this layer or above it. Consequently, when computing the attenuation factor we were limited to the case when one point is high above the inversion layer and the other is within the layer at a height equal to one-fifth the height of the inversion point.

2. ON THE HORIZON CONCEPT IN THE PRESENCE OF A TROPOSPHERIC WAVEGUIDE NEAR THE EARTH

Let us consider in more detail the horizon concept when a waveguide (inversion layer) exists near the earth.

First, let us recall the ray treatment of normal and anomalous propagation (see (3), pp. 16, 17). The reduced index of refraction is a linear function of the height for a homogeneous atmosphere.

The rays, issuing from the source Q, have the shape of curves inverted convexly to the s axis (Fig. 1a) on the s, h plane (s is the distance along the earth, h is the height). The horizon OO' is determined by the ray QOO' which touches the earth at the point O. To the right of the horizon line OO' is the shadow region which the field penetrates only because of diffraction; to the left is the line-of-sight region. The reflection formula, according to which the field is obtained as a result of the interference of the direct ray QP with the ray QP'P reflected from the earth, is approximately applicable for observation points in the line-of-sight region (to the left of the OO' horizon).

Rays from the source Q located within an atmospheric waveguide, near the earth, of height h_i (Fig. 1b) are convex upward (from the s

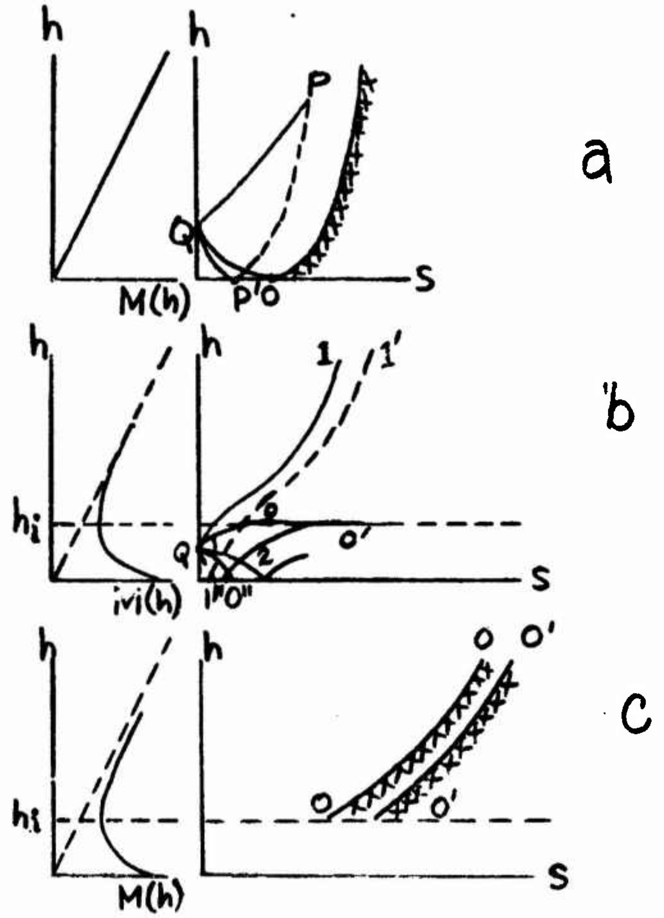

Fig. 1. To the horizon concept.
a - for normal refraction;
b - for superrefraction - according to geometric optics;
c - for superrefraction - according to wave optics

axis) within the waveguide and are convex downward (as in Fig. 1a) above the waveguide. Consequently, the ray Q1 passes into space above the waveguide but the ray Q2 appears to be 'trapped' within the waveguide. These two kinds of rays are separated by the limiting ray Q0 which approaches the height $h = h_1$ asymptotically as $s \longrightarrow \infty$. Besides the direct rays, rays reflected from the earth, as Q1''1' for example, are incident on the space above the inversion layer and are separated from the trapped rays by another limiting ray Q0"0' which approaches the height h_1 asymptotically after a single reflection from the earth. All ray issuing from a source within the angle 0Q0" formed by both the limiting rays appear to be trapped.

In this example, the laws of geometric optics lead to the conclusion that a horizon is absent both within and above the waveguide. Actually, direct rays issuing from Q within the angle 1Q0 and reflected rays issuing from within the angle 1"Q0" pass through observation points situated above the waveguide to the right of the rays 1 and 1'. They penetrate the whole space above the waveguide to the right of the rays 1 and 1' and, consequently, the region of geometric shadow and, therefore, the horizon are absent.

However, it is easy to see that the laws of geometric optics are not applicable to the limiting rays Q0 and Q0"0' and to rays close to the limiting. From the preceding, it is clear that precisely these rays transport (according to the geometric optics laws) electromagnetic energy to long distances above the waveguide. Hence, there follows that wave considerations must be drawn upon in order to solve the question of the horizon and the range of propagation with superrefraction.

This was done in (2) where it was shown that there is a certain boundary 0'0' (Fig. 1c) in the space above the waveguide, to the right of which a ray reflected from the earth cannot penetrate. This boundary 0'0' is the horizon in the presence of an inversion layer since to the right of this boundary, i.e., in the shadow region, the field (as in Fig. 1a) can only penetrate because of diffraction.

Besides the boundary 0'0', there is still the boundary 00, to the right of which direct rays which do not experience reflection from the earth, cannot penetrate. The boundary 0'0' is to the right of the boundary 00 since a ray, when reflected from the earth, appears to be to the right of a direct ray parallel thereto (see the rays Q1 and Q1"1' on Figure 1b). Direct rays do not pass into the 00 - 0'0' band, consequently, the total field in this band is not subject to the ray treatment. The total electromagnetic field to the left of the boundary 00 is obtained by the superposition of the direct and reflected rays.

Because of such a value for the boundary 00 - the limits of applicability of the reflection formula - it is expedient to introduce a special designation for it: we call it the direct wave horizon. In contrast, we call the boundary 0'0' the reflected wave horizon. While these horizons coincide for normal propagation, they must be differentiated in the case of anomalous propagation.

The horizons O'O' and OO on Fig. 1c replace the limiting rays QO"O' and QO (Fig. 1b), obtained from geometric optics, in the wave picture.

These general considerations will be made more precise in Section 4.

3. FUNDAMENTAL FORMULAS

The attenuation factor V in an inhomogeneous atmosphere for which the refractive index depends only on height can be represented as the contour integral:

$$V(x,y',y) = \exp\left(-i\frac{\pi}{4}\right)\frac{\overline{x}}{\pi}\int_C e^{ixt}\,F(t,y'y)\,dt \tag{1}$$

When an inversion layer is present near the earth, if one of the corresponding points is above the layer and the other is within it, then the following approximate expressions (see (2), Section 4) can be taken for the integrand F:

$$F(t,y'y) = \frac{\exp\left\{i\left[S(y') - 2S_0\right]\right\}\sin S(y)}{\sqrt{p(y') - t}\,\sqrt{p(y) - t}\,\left[\chi(\nu)\exp(-2iS_0) - ie^{-\pi\nu}\right]} \tag{2}$$

Here y' and y are the nondimensional heights of the source and the observation point ($y' > y$, where $y' > y_1$ and $y < y_1$, where y_1 is the nondimensional height of the inversion point); x is the nondimensional distance between the source and the observation point and p(y) is a function related to the reduced refractive index M(h) by the formula:

$$p(y) = \frac{2m^2}{10^6} M(h) = 2m^2\left(n - 1 + \frac{h}{a}\right); \qquad m = \left(\frac{ka}{2}\right)^{1/3} \tag{3}$$

where n is the refractive index of air; a is the radius of the earth.

We assume that the function M(h) has the same shape as on Figs. 1b and 1c. Consequently, for given t, the equation

$$p(y) - t = 0 \tag{4}$$

has two roots y_1 and y_2. These roots are real and positive for $p(y_i) < t < p(0)$; they are complex conjugates for $t < p(y_i)$; they coincide for $t = p(y_i)$ and then $y_1 = y_2 = y_i$. In general, there can be other roots (negative or complex) besides these two but they are of no value.

The quantities $S(y)$, $S(y')$ and S_o are given by the formulas:

$$S(y) = \int_0^y \sqrt{p(y) - t}\, dy; \qquad S(y') = \int_0^{y'} \sqrt{p(y) - t}\, dy$$

$$S_o = \frac{1}{2}\int_0^{y_1} \sqrt{p(y) - t}\, dy + \frac{1}{2}\int_0^{y_2} \sqrt{p(y) - t}\, dy \tag{5}$$

wherein the radical $\sqrt{p(y) - t}$ must be taken in the arithmetic sense for positive real y for $t < p(y_i)$. In order to evaluate S_o for $t < p(y_i)$, the radical $\sqrt{p(y) - t}$ must be continued analytically into the region of complex y. We will consider that p(y) is an analytic function (see Formula (18) below) admitting of such a continuation.

The quantity ν is determined from the formula:

$$\nu = \frac{1}{i\pi} \int_{y_1}^{y_2} \sqrt{p(y) - t}\, dy \tag{6}$$

The quantity ν is also real for real values of t, where the sign of ν is chosen from the following relations. The function p(y) can be replaced, for $y \approx y_2$, by the first terms of the Taylor series

$$p(y) = p(y_1) + \frac{1}{2} p'' (y_1)(y - y_1)^2$$

and the integral (6) can, afterward, be calculated and we obtain the following approximate formula for $t \approx p(y_1)$

$$\nu = \frac{p(y_1) - t}{\sqrt{2p''(y_1)}} \tag{7}$$

In conformance with this, we consider $\nu > 0$ for $t < p(y_1)$ and $\nu < 0$ for $t > p(y_1)$. Formula (6) is rewritten thus for $p(y_1) < t < p(0)$:

$$\nu = -\frac{1}{\pi} \int_{y_1}^{y_2} \sqrt{t - p(y)}\, dy \tag{8}$$

where $\sqrt{t - p(y)} > 0$ and $y_1 < y_2$.

The function $\chi(\nu)$ is determined by the formula:

$$\chi(\nu) = \frac{\sqrt{2\pi} \exp\left[-\frac{\pi}{2}\nu + i(\nu - \nu \ln \nu)\right]}{\Gamma\left(\frac{1}{2} - i\nu\right)} \tag{9}$$

where the principal value is taken for $\ln \gamma$ at $\gamma > 0$ $[t < p(y_t)]$.

Hence

(10) $$\chi(\gamma) \longrightarrow 1 \quad \text{as} \quad \gamma \longrightarrow \infty$$

When evaluating the attenuation factor for large values of y, it is necessary to take into account that the function p(y) must satisfy the following relation as $y \longrightarrow \infty$:

(11) $$\lim_{y \to \infty} [p(y) - y] = 0$$

Consequently, representing S(y') as follows:

$$S(y') = \int_0^{y'} \sqrt{y - t}\, dy + \int_0^{y'} \left[\sqrt{p(y) - t} - \sqrt{y - t}\right] dy$$

we see that the first component increases without limit as $y' \longrightarrow \infty$ (the infinite part equals $\frac{2}{3} y'^{3/2} - t\sqrt{y'}$) and the second tends to a finite limit if the difference p(y) - y approaches zero rapidly enough (for example, just as for the function p(y) according to (18) .

Let us introduce the quantity ξ_c as the limit

(12) $$\xi_o = \lim_{y \to \infty} \left[S(y') - 2S_c - \frac{2}{3} y'^{3/2} + t\sqrt{y'}\right]$$

Substituting the following for large values of y'

$$S(y') - 2S_o = \frac{2}{3} y'^{3/2} - t\sqrt{y'} + \xi_o$$

and replacing the quantity $\sqrt[4]{p(y') - t}$ in the denominator of (2) by $\sqrt[4]{y'}$, we obtain the attenuation factor as

(13) $$V(x,y',y) = \sqrt[4]{\frac{x^2}{y'}}\ \exp\left[i\,\frac{2}{3}\right] y'^{3/2}\ V_1(\zeta,y)$$

where

(14) $$V_1(\zeta,y) = \frac{\exp\left[-i\,\frac{\pi}{4}\right]}{\sqrt{\pi}}\int_C e^{i\zeta t}\,\Psi(t,y)\,dt$$

and

(15) $$\psi(t,y) = \frac{\exp(i\xi_0)\sin S(y)}{\sqrt{p(y)} - t\left[\chi(\nu)\exp(-2iS_0) - i\,e^{-\pi\nu}\right]}$$

The function $V_1(\zeta,y)$ is related to the attenuation factor V by the same formula (13) as in the theory of normal radiowave propagation. Just as in this latter theory, it is natural to call V_1 the attenuation factor of plane waves. Since we shall evaluate only V_1 subsequently, we shall often designate V_1 as simply the attenuation factor.

Let us introduce the variable ζ which equals

(16) $$\zeta = x - \sqrt{y'}$$

into V_1. The geometrical meaning of ζ follows from Fig. 2, where T denotes the point at which the incident plane wave (or spherical wave from a remote source) touches the earth's surface. The quantity ζ is related to the angle θ = TCP (P is the observation point, C is the center of the earth) or with the distance $s = a\theta$ along the earth which corresponds to it, by means of the relations

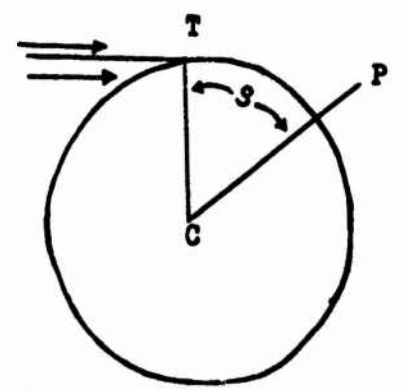

Fig. 2. Geometric meaning of ζ.

$$\zeta = m\theta = m\frac{s}{a}\;; \qquad m = \left(\frac{ka}{2}\right)^{1/3} \tag{17}$$

Let us note that the point of tangency T corresponds to the path of a ray in a homogeneoue atmosphere.

The infinite contour C in the plane of the complex variable t, over which the integrals for V and V_1 are taken, is arbitrary to a considerable degree and should be chosen so that the integral can be evaluated with the least difficulty, particularly, in such a way that the principal part of the integration would be as small as possible. Hence, the contour C should encircle all the poles of the integrand in a positive direction so that they would be above the contour C. It would appear to be more convenient to take the contour shown in Fig. 3, with its break-point either at $t = p(y_1)$ or somewhat to the left (see the end of Section (6) as the contour of integration.

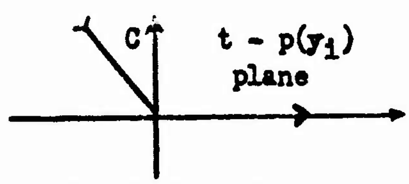

Fig. 3. Contour C in the complex $t - p(y_1)$ plane.

As is seen from (5) and (6), integrals of the form $\int \sqrt{p(y) - t}\; dy$ for different t and for different limite of integration, including the complex, enter into the integrand $\Psi(t,y)$. In order to facilitate the evaluation of these integrals, the hyperbolic law (54) wae taken for the reduced index of refraction M(h) and, consequently, the function p(y) is obtained according to (3) as

$$p(y) = p(y_1) + \frac{(y - y_1)^2}{y + y_2} \tag{18}$$

whereupon there results from (11) that

(19) $$p(y_1) = y_2 + 2y_1$$

Two parameters, y_1 and y, are in (18), where y_1 is the nondimensional height of the inversion point. It is also expedient to introduce the special notation

(20) $$Y = y + y_1$$

then

(21) $$p''(y_1) = \frac{2}{Y}$$

Let us note that (4) is a quadratic equation with the two roots y_1 and y_2, which join at $t = p(y_1)$, in the case of the hyperbolic law.

The integrals which we need in the case of the hyperbolic law are expressed through elliptic integrals of the first and second kinds. However, in the cases we considered, it appeared to be more convenient to evaluate these integrals by expansion in powers of the parameter a^2, where

(22) $$a^2 = \frac{t - p(y_1)}{4Y}$$

It is sufficient to take several of the first terms in these expansions, which also contain logarithmic components, since the principal part of the integration over C corresponds to very small values of the parameter a^2. Later terms of the expansion are essential for the large values of the parameter Y which we took (see the beginning of section 5) only on those parts of the contour where the whole integrand is itself small.

In conclusion, let us dwell on the analytic continuation of the functions $F(t,y',y)$ and $\Psi(t,y)$ over the whole complex t plane. The fact is that the quantities $S(y)$, $S(y')$, S_0 and $\chi(\nu)$, which enter into these functions, are originally defined only on the real axis for $t < p(y_1)$ $(\nu > 0)$ where the arithmetic values were taken for the radicals $\sqrt[4]{p(y') - t}$ and $\sqrt[4]{p(y) - t}$. However, knowing the integrand at t $p(y_1)$ appears to be sufficient only for calculations with the reflection formula (see Section 4). The integrands must be known for complex t in order to calculate the contour integrals, and this is accomplished by using analytic continuation.

Here, it must be kept in mind that the exact functions $F(t,y',y)$ and $\Psi(t,y)$ have no singularities at the point $t = p(y_1)$. However, the asymptotic expression (15) for the function $\Psi(t,y)$ has a singular point (a branch point) at $t = p(y)$ (for expressions $\sqrt[4]{p(y) - t}$ and for $S(y)$) and at $t = p(0)$ (for $S(y)$ and ξ_0). These singular points are obtained because we used the asymptotic expressions. Actually, there are no branch points since the exact integrand must be meromorphic. Consequently, we bypass the 'apparent singular points' from below by considering, for example, that $\arg[p(y) - t] = \pi$ for $t > p(y)$ and that $\sqrt{p(y) - t} = i\sqrt{t - p(y)}$, where $\sqrt{t - p(y)} > 0$. In substance, this bypass is conditional since (2) is not applicable for $t > p(y)$ because of the so-called Stokes phenomenon. This phenomenon can only be neglected when the section $t > p(y)$ gives a small contribution to the value of the contour integral, as occurs in the cases which we consider. The check calculations which we made by using parabolic cylinder functions (see [2], Section 3), which give a more exact asymptotic representation of the integrand $\Psi(t,y)$, confirmed both the qualitative and the quantitative validity of the results obtained by using (15).

The function $\Psi(t,y)$ also has poles corresponding to the roots of Equation (45). When the poles approach close to the contour of integration, they must be bypassed from below.

4. REFLECTION FORMULA

It is natural to evaluate the attenuation factor in the line-of-sight region by the method of stationary phase since this method gives the transition to the laws of geometric optics which is applicable far enough from the horizon. The method of stationary phase can be applied to the integral of (14) as follows. Let us represent the integrand on the real axis as:

$$\Psi = \frac{i}{2} \frac{e^{i\Omega(t)} - e^{i\Phi(t)}}{\sqrt[4]{p(y) - t}\,|X(\nu)|\,(1 - \Lambda)} \tag{23}$$

where

$$\begin{aligned} \Omega(t) &= \xi_0 - S(y) + 2S_0 - \arg X(\nu) \\ \Phi(t) &= \xi_0 + S(y) + 2S_0 - \arg X(\nu) = \Omega(t) + 2S(y) \end{aligned} \tag{24}$$

$$-\arg X(\nu) = \nu \ln \nu - \nu + \arg\Gamma\left(\frac{1}{2} - i\nu\right) \tag{25}$$

and

(26) $$\Lambda = \frac{1}{\chi(\nu)} \exp\left[-\pi\nu + 2iS_o\right]$$

The following expression can be written for all the integrands in (14) for real t:

(27) $$e^{i\zeta t}\,\Psi = \frac{1}{2}\,\frac{e^{i\omega(t)} - e^{i\varphi(t)}}{\sqrt{p(y) - t}\,|\chi(\nu)|\,(1 - \Lambda)}$$

where

(28) $$\begin{cases} \omega(t) = \zeta t + \Omega(t) \\ \varphi(t) = \zeta t + \Phi(t) \end{cases}$$

Since ν is also real here, then

(29) $$|\chi(\nu)| = \sqrt{1 + e^{-2\pi\nu}}$$

and if $\nu > 0$, then

(30) $$|\Lambda| = \frac{1}{\sqrt{1 + e^{2\pi\nu}}}$$

The last formula shows that the absolute value of Λ is less than unity (in particular, $|\Lambda| = \frac{1}{\sqrt{2}}$ for $\nu = 0$) if $\nu > 0$ $[t < p(y_1)]$ and it tends rapidly to zero as ν increases. Consequently, if we should seek the stationary phase point at $t < p(y_1)$, we can neglect the phase of the denominator $1 - \Lambda$. Then the stationary phase points t_1 and t_2 of the first and second components in the right side of (27) are obtained from the equations

(31) $$\omega'(t_1) = 0\,; \quad \varphi'(t_2) = 0$$

or

$$(32) \qquad \zeta = -\Omega'(t_1); \quad \zeta = -\Phi'(t_2)$$

where the values of t_1 and t_2 are different for given ζ and y.

Calculations show that the functions $-\Omega'(t)$ and $-\Phi'(t)$ have a maximum. Consequently, we find two values of t_1 and two values of t_2 (at least, if ζ is not too large). Only values of t_1 and t_2 should be taken which correspond to the left half of the curves $t < p(y_1)$ $(a^2 < 0)$ inasmuch as the phase of the denominator $1 - \Lambda$ can be neglected only for these values when determining the stationary phase points.

Finding the points t_1 and t_2, we can evaluate (14) by applying the method of stationary phase to each component of (27). Thus, we arrive at the reflection formula for the attenuation factor V_1:

$$(33) \qquad V_1(\zeta, y) = \frac{\exp[i\omega(t_1)] \quad A(t_1)}{\sqrt[4]{p(y) - t_1} \sqrt{-2\omega''(t_1)}} - \frac{\exp[i\varphi(t_2)] \quad A(t_2)}{\sqrt[4]{p(y) - t_2} \sqrt{-2\varphi''(t_2)}}$$

where

$$(34) \qquad A(t) = \frac{1}{|\chi(\nu)| \ (1 - \Lambda)}$$

The first term of the reflection formula (33) is the ground wave, the second term is the wave reflected from the earth. This formula has the same structure as the usual reflection formula of geometric optics, however, corrections, arising in the exact analysis of wave passage through a layer adjoining the inversion point, are reflected therein.

Let us note that t_1 and t_2 decrease as ζ decreases and the values of ν corresponding thereto increase. It is possible to write for large enough positive ν

(35) $$\Lambda = 0; \quad X(\nu) = 1; \quad A = 1$$

and, consequently, the more simple expressions can be used for the $\Omega(t)$ and $\Phi(t)$ functions

(36) $$\Omega(t) = \Xi_0 - S(y)$$
$$\Phi(t) = \Xi_0 + S(y)$$

where

$$\Xi_0 = \zeta_0 + 2S_0 = \lim_{y' \to \infty}\left[S(y') - \frac{2}{3}y'^{3/2} + t\sqrt{y'}\right]$$

The reflection formula (33), for such simplifications, transforms into the usual reflection formula resulting from the laws of geometric optics in an inhomogeneous atmosphere. Therefore, the latter is applicable to rays sufficiently far from the limiting rays QO and QO"O on Fig. 1b, more exactly, to those rays for which $\nu(t_1)$ and $\nu(t_2)$ are large enough positive numbers. As it is easy to consider, we have $\nu = 0$ for the limiting rays themselves and geometric optics is not applicable to them.

Returning to the general reflection formula (33), let us introduce the following notation for the maximum values of $-\Omega'(t)$ and $-\Phi'(t)$:

(38) $$\zeta_1 = [-\Omega'(t)]_{max}; \quad \zeta_2 = [-\Phi'(t)]_{max}$$

Because of (24), the following inequality is always satisfied:

(39) $$\zeta_1 < \zeta_2$$

Hence, we see that the stationary phase point t_1 and t_2 can only be found for both components in (27) if $\zeta < \zeta_1$. The equation $\omega'(t) = 0$ has no real solution for $\zeta > \zeta_1$ and the ground wave is not expressed by the first component of (33). Consequently, the value $\zeta = \zeta_1$ determines the horizon of the ground waves (see Section 2). Similarly, the value $\zeta = \zeta_2$ determines the horizon of the waves reflected from the earth.

The physical meaning of ζ_2 is that the electromagnetic waves escape in the $\zeta > \zeta_2$ region only because of diffraction, consequently, $\zeta = \zeta_2$ is the boundary of the shadow region. The physical meaning of ζ_1 is that reflection formula (33) is applicable for $\zeta < \zeta_1$, consequently, $\zeta = \zeta_1$ is the boundary of the line-of-sight region. The region $\zeta_1 < \zeta < \zeta_2$ is the intermediate region between both horizons.

Since the maximum values of the functions $-\Omega'(t)$ and $-\Phi'(t)$ are attained near the point $t = p(y_1)$, then the quantities

$$(40) \qquad \tilde{\zeta}_1 = \left[-\Omega'(t)\right]_{t=p(y_1)} \; ; \qquad \tilde{\zeta}_2 = \left[-\Phi'(t)\right]_{t=p(y_1)}$$

will be very close to the quantities determined by (38), as we will show by examples in Section 5. Consequently, the location of the horizon can be determined approximately by a formula such as (40), which is much more simple than to construct the graphs of the functions $-\Omega'(t)$ and $-\Phi'(t)$ which are required for the use of (38). The formulae (40) for the hyperbolic law (18) reduce to

$$(41) \qquad \tilde{\zeta}_1 = G_0 - G(y) \; ; \quad \tilde{\zeta}_2 = G_0 + G(y)$$

where

$$(42) \qquad G_0 = \sqrt{y_\ell} - \frac{\sqrt{Y}}{2} \ln \frac{\sqrt{Y} + \sqrt{y_\ell}}{\sqrt{Y} - \sqrt{y_\ell}} + \frac{\sqrt{Y}}{2}\left[C_1 + \frac{1}{2}\ln(Y^3)\right]$$

$$(43) \qquad G(y) = -\sqrt{y_\ell + y} + \sqrt{y_\ell} + \frac{\sqrt{Y}}{2}\left[\ln \frac{\sqrt{Y} + \sqrt{y_\ell + y}}{\sqrt{Y} - \sqrt{y_\ell + y}} - \ln \frac{\sqrt{Y} + \sqrt{y_\ell}}{\sqrt{Y} - \sqrt{y_\ell}}\right]$$

and

$$(44) \qquad C_1 = C + 7 \ln 2 - 4 = 1.429$$

(C is Euler's constant).

The second formula of (41) gives (115) of [2] for the distance of the horizon of the reflected waves when the transformation is made to the usual (dimensional) coordinates. As we already said, the first formula determines the distance of the ground wave horizon.

In conclusion, let us note that reflection formula (33) is applicable to the calculation of the attenuation factor V_1 almost up to the ground wave horizon Γ_1 itself.

5. NUMERICAL RESULTS IN NONDIMENSIONAL COORDINATES

We chose the following numerical values of the parameters which enter into the function $p(y)$ [Formulas (18)-(20)] when calculating the attenuation factor V_1 for a hyperbolic inversion law:

TABLE 1

No.	y_1	y	Y	$p(y_1)$	$p(0)-p(y_1)$	y
1	10.40	19[illegible].61	208.01	218.41	0.542	2.08
2	5	95	100	105	0.260	1
3	2.40	45.67	48.07	50.48	0.125	0.48
4	1.16	21.95	23.11	24.27	0.060	0.23

The functions p(y) for the values of the parameters selected are

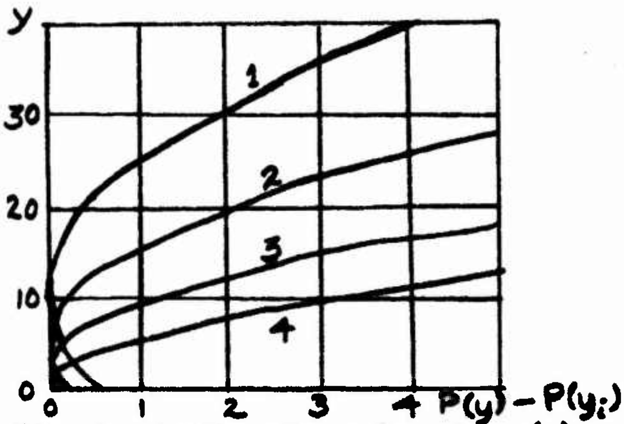

Fig. 4. Graphs of the function p(y) for the parametric values of Table 1.

shown in Fig. 4. The choice we made permits the propagation of four wavelengths, which are referred to as 1:3:9:27, to be calculated for a specific M-profile (see Section 7). Here the first row of Table 1 corresponds to very short waves and the fourth row corresponds to very long waves.

We took $y = \frac{y_i}{5}$ in all cases, i.e., we assumed the height of one of the corresponding points to be equal to one fifth the height of the inversion layer. We took the other point at a great height above the inversion layer - so great a height that the attenuation factor $V_1(\zeta, y)$ of (13) could be used.

The four curves of the attenuation factor V_1, which we calculated as a function of the variable ζ, are given on Fig. 5. The subscripts

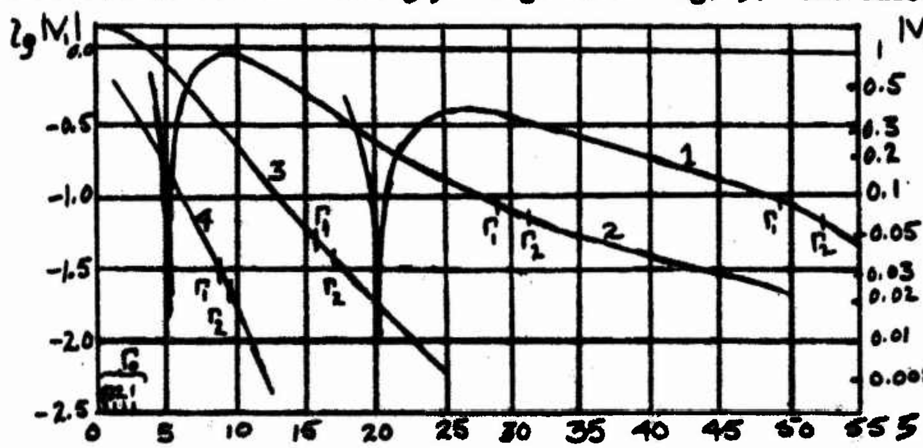

Fig. 5. Dependence of the attenuation factor V_1 on ζ

Curves 1,2,3,4 correspond to the numbers of the rows of Table 1 and to the numbers of the curves of Fig. 4.

1,2,3, and 4 on the curves show to which row of Table 1 and to which p-curve of Fig. 4 the given curve for the attenuation factor corresponds. The point Γ_1 on each curve marks the location of the ground wave horizon and the point Γ_2 marks the location of the horizon of waves reflected from the earth. The points Γ_0 near the origin, which are provided with the same subscripts 1, 2, 3, and 4, determine the horizon (the line-of-sight limit) for a homogeneous atmosphere; the corresponding values ζ_0 are obtained from the simple formula $\zeta_0 = \sqrt{y}$.

As is seen, long distance propagation occurs in all four of the cases considered, and as should be expected, the most sharply expressed is in Curve 1. The phenomenon of long-distance propagation attenuates monotonically when the transition is made to Curves 2, 3, and 4, however, the attenuation factor $|V| \approx 0.1$ for $\zeta \approx 5$ according to Curve 4 while $|V_1|$ assumes a value four orders lower ($|V_1| \approx 0.000013$) for the same ζ and y but in a homogeneous atmosphere.

The values of the function $|V_1|$ at the horizons Γ_1 and Γ_2 are given in Table 2.

It is seen therefrom that the values of the attenuation factor at both horizons Γ_1 and Γ_2 vary within sufficiently wide limits, from 3 - 3.5 times. The values of $|V_1|$ at the Γ_0 horizon for normal propagation and for the same values of y are given for comparison on Table 2. A comparison of the columns shows that the values of the attenuation factor at the horizon for normal propagation have approximately the same scatter as for anomalous propagation to the Γ_1 and Γ_2 horizons, because of the dependence on y.

Table 2

No.	Γ_1	Γ_2	Γ_o	y
1	0.096	0.070	0.24	2.08
2	0.095	0.080	0.19	1
3	0.047	0.035	0.14	0.48
4	0.031	0.023	0.083	0.23

Table 3

No.	ζ_1	$\tilde{\zeta}_1$	ζ_2	$\tilde{\zeta}_2$
1	49.11	49.11	52.26	52.26
2	28.56	28.56	30.74	30.74
3	16.08	15.99	17.52	17.50
4	8.67	8.45	9.52	9.50

It can be noted that a sudden variation in the character of the propagation does not occur at the Γ_1 and Γ_2 horizons: The attenuation factor starts to decrease monotonically in the line-of-sight region to the left of both horizons. In particular, this leads to the attenuation factor being 2 - 4 times less at the Γ_1 and Γ_2 horizons, according to Table 2, than at the Γ_o horizon for normal propagation. Such a behavior of the attenuation factor is apparently explained by diffraction (more accurately, wave) phenomena, taken into account by the reflection formula (33) and not included in the laws of geometric optics, having value not only beyond the Γ_1 and Γ_2 horizons but to the left as well.

In order to explain the applicability of the simple formulas (40)-(44) to compute the distances of the Γ_1 and Γ_2 horizons, let us compare the results which they give in the cases we considered with the results obtained from formula (38).

Table 3 shows that both formulas give very close numbers. Consequently, the simple formulas of (2) can be used to compute the distance to the horizons in practical computations.

6. ATTENUATION FACTOR IN DEEP SHADE. RESIDUE SERIES

It is convenient to investigate the attenuation factor in deep shade by using the residue series which is obtained from the integral (14) by the usual method [see [1], Section 6]. In order to obtain the residue series, it is first necessary to obtain the exact location of the poles of the function $\Psi(t,y)$, i.e., the roots of the equation

$$1 - \Lambda = 0 \tag{45}$$

These roots are found near the contour C (Fig. 3) or within it. If we denote

$$\Delta t = t - p(y_1) \tag{46}$$

then the values of Δt for the roots which we found form Table 4 in which the first column shows the number of the row in Table 1 and the second column shows the number of the root for this case.

Table 4

No.	m	Δt_m
1	1	0.10653 + i 0.00019
	2	-0.06364 + i 0.05523
	3	-0.1633 + i 0.2107
	4	-0.2495 + i 0.3913
2	1	-0.06338 + i 0.06518
	2	-0.1733 + i 0.3293
3	1	-0.1038 + i 0.2238
	2	-0.1883 + i 0.6934
4	1	-0.0852 + i 0.4661
	2	-0.1275 + i 1.1318

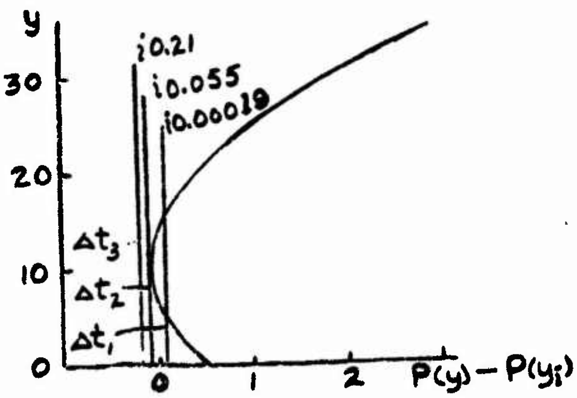

Fig. 6. Roots t_m corresponding to trapped and untrapped waves.

The location of the real parts of the first three roots of the p(y) curve is shown on Fig. 6 for the first case. We see that only the first root corresponds to the 'trapped' wave in the usual interpretation, the other two roots yield waves which easily emerge beyond the limits of the inversion layer, from the geometrical optics viewpoint. However, these 'leakage' waves have slight attenuation and participate actively in the very-long propagation process. Let us recall that $t_1 = 1.17 + i2.02$ for normal propagation so that the third wave attenuates ten times more slowly in this case than the least attenuated wave under normal propagation conditions. All the roots correspond to the 'leakage' waves for the rest of the cases.

Let us transform (45) to a simple approximate form which will permit comparison with other very-long propagation theories. Let us start with the 'trapped' waves which have almost real t between $p(y_1)$ and $p(0)$ (such as the first root in Table 4) and, therefore, have negative values of ν. For $\nu > 0$, we put

$$\nu = (-\nu)e^{i\pi} \; ; \quad \ln \nu = \ln (-\nu) + i\pi \tag{47}$$

Then we will have in addition to (10)

$$\chi(\nu) \longrightarrow 1 \quad \text{as} \quad \nu \longrightarrow -\infty \tag{48}$$

and we obtain from (5)

$$s_o = s_1 - \frac{i\pi}{2}\nu \tag{49}$$

where

$$s_1 = \int_0^{y_1} \sqrt{p(y) - t}\, dy \tag{50}$$

and y_1 denotes the least positive root of (4). Taking these formulas into account (45) becomes

$$i \exp(2iS_1) = \chi(\nu) \tag{51}$$

If ν is large and negative (strongly trapped waves) then we obtain the following more simple equation because of (48)

$$S_1 = (m - \frac{1}{4})\pi\,; \quad m = 1,2,\ldots \tag{52}$$

which corresponds to the known characteristic equation of trapped waves (see [3], p. 20).

Now, let us imagine that ν is positive or complex with positive real part, i.e., Re $t < p(y_1)$ or Re $\Delta t < 0$. In this case, it is not possible to determine the quantity S_1 by using (50), if only because it is not known which of the complex roots y_1 and y_2 should be taken. However, inverting (49), we can always determine S by using the relation:

$$S_1 = S_0 + \frac{i\pi}{2}\nu \tag{53}$$

and we again obtain (51) from (45). Since we will always have $\chi(\nu) \to 1$ for a suitable choice of arc ν [as in (47)] and for $|\nu| \to \infty$ with the exception of arc $\nu = -\frac{\pi}{2}$ and, moreover, $\chi(0) = \sqrt{2}$, then we can consider $\chi(\nu) = 1$ as a first, quite rough approximation for the 'leakage' wave and we again obtain (52).

Let us note that the simplified equation (52) is also suitable for normal propagation when it is necessary to put $p(y) = y$ and $y_1 = t$ in (50). We thus obtain from (52)

$$t_m = \left[\frac{3}{2}\left(m - \frac{1}{4}\right)\pi\right]^{2/3} \exp\left(\frac{i\pi}{3}\right) \tag{54}$$

which corresponds approximately to the roots of the characteristic equation for the homogeneous atmosphere.

In order to verify (52), we calculated the value of S_1 for the roots which we found according to (53) and we obtained the following numbers as a result:

Table 5

No.		$(m - \frac{1}{4})\pi$	S_1	ν
1	1	2.356	2.326 - i 0.001	-0.768 - i 0.0014
	2	5.498	8.537 + i 0.047	0.459 - i 0.398
	3	8.639	9.646 + i 0.009	1.178 - i 1.519
	4	11.781	11.784 + i 0.005	1.800 - i 2.821
2	1	2.356	2.444 + i 0.062	0.317 - i 0.376
	2	5.498	5.501 + i 0.011	0.867 - i 1.646
3	1	2.356	2.315 + i 0.011	0.360 - i 0.776
	2	5.498	5.516 + i 0.014	0.656 - i 2.402
4	1	2.356	2.436 + i 0.079	0.207 - i 1.120
	2	5.498	5.499 + i 0.007	0.319 - i 2.718

Hence, by calculating S_1 for the roots found, we can ascribe the subscript m to it by using the approximate relation (53).

Shown on Fig. 7 is the attenuation factor in deep shade calculated by using the residue series for the first case. Figure 7 shows that the first term of the residue series, which corresponds to the pole t_1, only determines the attenuation

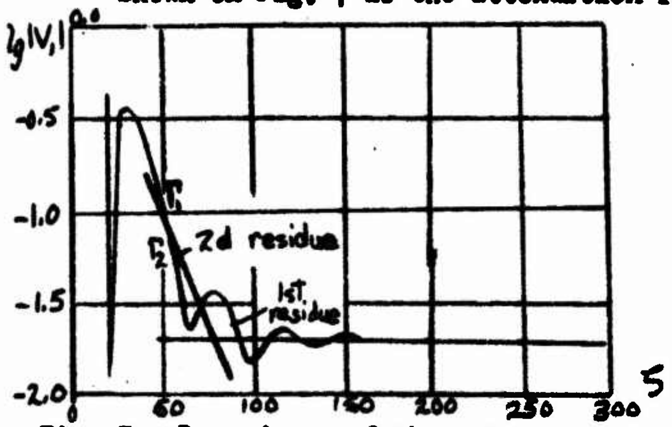

Fig. 7. Dependence of the attenuation factor V_1 on ς in the deep shade as calculated using the residue series.

factor for $\zeta > 150$, i.e., for λ = cm waves at $s > 1000$ km. Since the first term has negligible attenuation, then the absolute value of the attenuation factor will be almost constant at such long ranges, the asymptote on Fig. 7 is almost horizontal. Let us note that the attenuation factor approaches the asymptote completing the attenuation of the oscillations in the deep shade on Fig. 7. These oscillations caused the interference of the first and second 'simple waves'.

Hence, the first simple wave with the least attenuation is excited very slightly by a wave incident from above onto the tropospheric waveguide because of which this simple wave can have a decisive value only at very long ranges. The second and the third, in part, terms of the residue series have fundamental value near the Γ_1 and Γ_2 horizons. This phenomenon must have a general character since if the simple wave is 'trapped' (see above) and almost does not leak out of the inversion layer (which is explained by its negligible attenuation) then it is almost not excited by radiators above the inversion layer according to reciprocity considerations. Waves with large attenuation to a large degree penetrate the space above the inversion layer, consequently, they are excited more strongly and play a fundamental part near the horizons.

Because of the circumstance noted, the Γ_1 and Γ_2 horizons actually determine (although in an approximate enough sense) long distance radiowave propagation even for strongly expressed superrefraction, as is seen from Fig. 7.

The residue series is usually used as the basis for analyzing very long propagation. Here it is assumed that only trapped waves can have low attenuation (Re $\Delta t_m > 0$). Actually, waves which 'leak' also attenuate slightly in a number of cases (Re $\Delta t_m < 0$). Consequently, waves which are several times longer than the 'critical' wavelength λ_0 defined according to [4] (p. 258) contribute to very long propagation in a tropospheric waveguide.

In conclusion, let us note that several of the first terms of the residue series, as computations showed, permit the attenuation factor to be calculated until it almost joins the reflection formula and, hence, gets rid of calculations in quadratures (see Section 3).

7. NUMERICAL RESULTS FOR A CONCRETE CASE

In order to facilitate the physical analysis of the numerical results which we obtained in Section 5, we consider the corresponding concrete case herein.

The M-profile shown on Fig. 8 can be taken as an example and the attenuation factor V_1 can be constructed for the following wavelengths: 1) 3.33 cm; 2) 10 cm; 3) 30 cm; 4) 90 cm as is done in Fig. 9. The numbers on the curves of Fig. 9 indicate the wavelengths listed here.

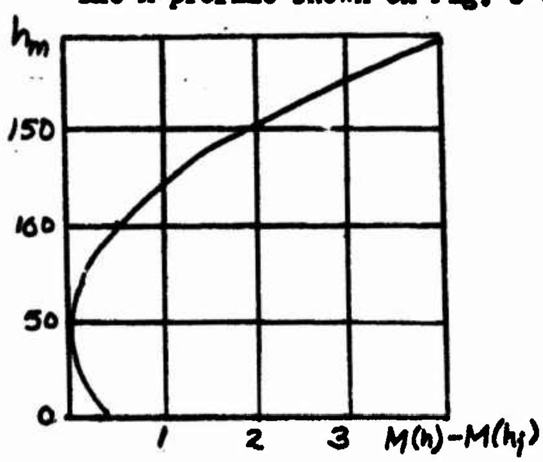

Fig. 8. Dependence of h_M on M (M - profile)
h_1 = 46.5 m; l = 884.0 m; H = 930.5
$M(h_1)$ = 153.5; $M(0) - M(h_1)$ = 0.381.

The lower horizontal scale is the range s in kilometers and the upper is the angle θ in degrees (see Fig. 2). The left vertical scale is the $\lg|V_1|$ (to the base 10) and the right hand scale is for values of $|V_1|$.

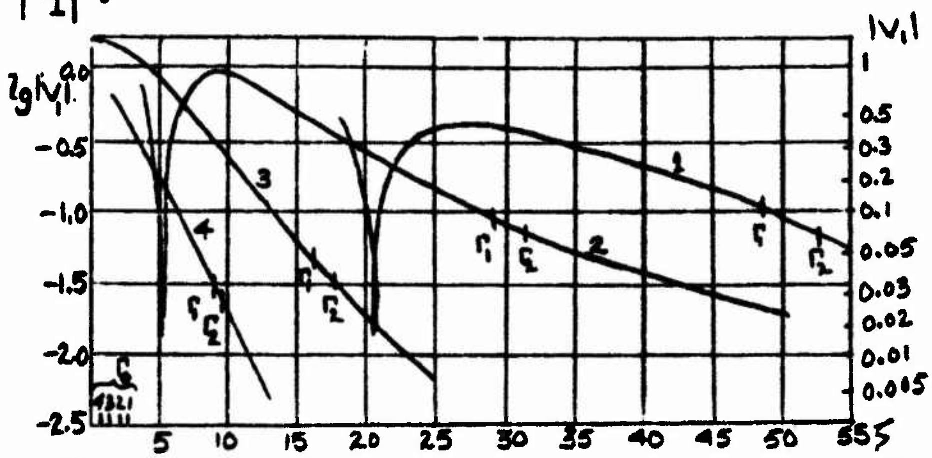

Fig. 9. Dependence of the attenuation factor V_1 on ς for the wavelengths: 1 - 3.33 cm; 2 - 10 cm; 3 - 30 cm; 4 - 90 cm.

Let us note that the dispersion was not taken into account in our computations. We assume that the M curve has the same shape for all four wavelengths for which the attenuation factor V_1 is given on Fig. 9.

The M curve on Fig. 8 is constructed according to the hyperbolic law

(55) $$M(h) = M(h_1) + \frac{1}{a} \frac{(h - h_1)^2}{h + l}$$

in which

(56) $$M(h_1) = \frac{l + 2h_1}{a}$$

Two parameters are included in the hyperbolic law: h_1 and l with the dimensionality of a height and related to the nondimensional constants y_1 and y in (18) by means of the relations

$$(57)\qquad y_i = \frac{kh_i}{m};\quad y_l = \frac{kl}{m};\quad m = \left(\frac{ka}{2}\right)^{1/3}$$

in which h_i is the height of the inversion point or, what is the same, the height of the atmospheric waveguide. As is easily shown, the height

$$(58)\qquad H = h_i + l \quad \left(Y = \frac{kH}{m}\right)$$

determines the radius of curvature of the M curve at the inversion point.

Also marked off along the horizontal axis of Fig. 9 is the horizon Γ_0 for propagation in a homogeneous atmosphere. This horizon is determined by the height of the observation point h and is independent of the wavelength: Let us note that we have taken $h = \frac{1}{5} h_i$ here. The point Γ_1 determines the position of the ground wave horizon on each curve and the point Γ_2 determines the position of the horizon for waves reflected from the earth (Sections 2 and 4). The Γ_1 and Γ_2 horizons vary as the wavelengths vary and, consequently, for each curve itself.

In all cases the phenomenon of very long radiowave propagation attenuated as the wavelength increases can be expressed. Taking into account the intense variation of the wavelength when making the transition from one curve to another (the wavelengths are in the 1:3:9:27 ratio), it should be recognized that the dependence of the attenuation factor on the wavelength is comparatively slight near the horizon.

The wavelength enters into the formula for the distance to the horizon (see [2], Section 5) only under the logarithm. Consequently, the distances to the horizons generate an arithmetic progression if the wavelengths, as in Fig. 9, generate a geometric progression. Here, however, the values of the attenuation factor at both the Γ_1 and the Γ_2 horizons depend on the wavelength to the same degree as for normal radiowave propagation (see Table 2).

Because of these circumstances, to identify the remoteness of radiowave propagation with the remoteness of the horison of the ground and reflected waves must be done with some care. The distance of propagation can be defined otherwise, for example, as that range in which the attenuation factor has the absolute value 0.1, where the values of the attenuation factor are still less at longer distances. For this last definition, the 'distance of propagation' is included between the distances of the Γ_1 and Γ_2 horizons for the Curve 1 on Fig. 9 and for the other curves, this distance is less than the distance Γ_1; as seen from the figure, these four distances generate an arithmetic progression in a very rough approximation. Let us note that it is usually sufficient to compute only by using the reflection formula of Section 4 and by extrapolating the curves thus obtained in order to estimate the distance of propagation according to the 0.1 value.

The direct purpose of this paper (see Section 1) was to verify the formulas for the distance of radiowave propagation derived in [2]. We have shown above that a simple and graphic picture of very-long radiowave propagation in the presence of an inversion layer can be obtained by introducing the horizons of the direct and reflected waves. However, the distance of propagation can only be identified with the distance of one of the horizons in only a sufficiently rough sense. The fact is that the decrease in the attenuation factor (after the oscillations terminate in the line-of-sight region) starts earlier than we arrive at the first horizon. Consequently, as shown in Section 5, the attenuation factor V_1 takes values on the Γ_1 and Γ_2 horizons which are 2 - 4 times less than at the usual Γ_0 horizon for propagation in a homogeneous atmosphere. Moreover, the attenuation factor decreases near the Γ_1 and Γ_2 horizons much more slowly, understandably, than for normal propagation.

All these causes reduce to the Γ_1 and Γ_2 horizons characterizing the distance of radiowave propagation more roughly for anomalous propagation than does the Γ_0 horizon under normal propagation. However, the possibility of using the Γ_1 and Γ_2 horizons for an approximate estimate of the distance of propagation does not cause doubts, as is seen if only from a comparison of the attenuation factors near the horizons and in deep shade on Fig. 7.

It should be stressed that the M-profile we chose has a weak enough inversion: the difference $M(0) - M(h_1)$ does not exceed several tenths. In certain cases, such an inversion can remain unestablished in practice. However, our calculations show that even such an M-profile radically alters the character of radiowave propagation and leads to very long propagation.

Leningrad University April 27, 1956

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p. 382 - Blank

Appendix A

APPROXIMATE BOUNDARY CONDITIONS FOR THE ELECTROMAGNETIC FIELD ON THE SURFACE OF A GOOD CONDUCTOR

M. A. Leontovich

The approximate boundary conditions on the surface of bodies which have a large complex permittivity have found application in solving a number of problems concerning the propagation of electromagnetic waves.* In view of the fact that a very detailed derivation of such boundary conditions has yet to be published, this present paper derives these boundary conditions and indicates the limits of their applicability.

1. As we know, the problem of the propagation of electromagnetic waves when "ideally conducting" bodies are present reduces to the solution of the problem involved in the propagation of a field outside these bodies under specific boundary conditions at the surface of these bodies (the tangential components of the $\vec{E}$ vector are equal to zero). The problems involving the propagation of a field outside good conductors (or, in general, outside bodies with a permittivity which has a large modulus) can also, under known conditions, be approximately reduced to the solution of the Maxwell equations for external (with respect to these bodies) space when homogeneous boundary conditions obtain at the surfaces of these bodies.

* Ia. L. Al'pert, Application to Losses in Waveguides. J. Tech. Phys. (No. 16) 10:1358, 1940.

The approximate boundary conditions being examined here are also given in a book by A. N. Shchukin titled "The Propagation of Radio Waves," 1940, p. 50, but the fully developed applications of these boundary conditions are not given.

If the complex dielectric (or magnetic) permeability of the body has a large modulus, then the wavelength inside the body (and in the case of an absorbing body the depth of penetration of the field into the body) will be small, so that inside the body the conditions obtain for the application of geometric optics. If, in addition to this, the field varies slowly from point to point on the surface of the body on the scale of a wavelength inside the body, and there are no sources inside the body, then the field in the vicinity of the surface (inside the body) will consist of a wave which is propagated and attenuated in the direction of the normal to the surface (into the interior) of the body. This wave, generally speaking, is not a plane wave, but its radius of curvature is large in comparison to the wavelength of the wave in the body and the depths of its penetration. Therefore to a first approximation the electric and magnetic vectors in the body are parallel to the surface of the wave; they lie in the plane which is tangent to the surface of the body and are related with one another in the same way that the electric and magnetic vectors are related in a plane wave (that is,

$$\vec{E} = \sqrt{\frac{\mu}{\epsilon}}\,[\hat{n}, \vec{H}]$$

where $\hat{n}$ is the external normal to the surface of the body; ϵ and μ are the complex electric and magnetic permeabilities of the body).

Since the tangential components of $\vec{E}$ and $\vec{H}$ are continuous, the tangential components of $\vec{E}$ and $\vec{H}$ are related by the same expression on the external side of the surface of the body; it therefore follows that the following boundary conditions are fulfilled there:

$$[\hat{n}, \vec{E}] = \sqrt{\frac{\mu}{\epsilon}}\,[\hat{n}, [\hat{n}, \vec{H}]]. \qquad (1)$$

Introducing the coordinate system (x, y, z) which is such that x and y lie in the plane which is tangent to the surface of the body at the point being investigated, and the z axis is directed into the interior of the body, it is possible to write these boundary conditions in the following manner:

$$E_x = \sqrt{\frac{\mu}{\epsilon}} H_y; \quad E_y = -\sqrt{\frac{\mu}{\epsilon}} H_x. \tag{2}$$

2. Our problem consists of making the limits of applicability for these boundary conditions more precise and of evaluating the errors which are associated with the use of these boundary conditions. We must therefore, in the first place, evaluate errors which are associated with the depiction of the field in the form of a wave which is propagated into the interior of the body according to the laws of geometric optics, and, in the second place, we must clarify under what conditions this field can be represented in the form of a wave of this type. The answer to the first problem is contained in a paper by S. M. Rytov.* We shall reproduce here the results ofhis paper in the form required for our analysis.

We shall examine a body with a complex permittivity ϵ, with a magnetic permeability μ, both of which vary from point to point in the body. Here we shall assume that the complex index of refraction $\sqrt{\epsilon\mu}$ is a quantity which has a large modulus everywhere in the body. Therefore, we assume that

$$\sqrt{\epsilon\mu} = \frac{v(x, y, z)}{q},$$

where q is a small parameter.

Having written the Maxwell equations:

$$-ik\epsilon\vec{E} = \mathrm{Curl}\,\vec{H}, \qquad ik\mu\vec{H} = \mathrm{Curl}\,\vec{E} \tag{3}$$

(k is the wave number in a vacuum, and the time function is assumed to be of the form $e^{-i\omega t}$) in the following form:

*S. M. Rytov., J. Expt. and Theor. Phys. (No. 2) 10; 180, (1940).

$$\left.\begin{aligned} -ik\sqrt{\epsilon\mu}\,(\sqrt{\epsilon}\,\vec{E}) &= \text{Curl}\,(\sqrt{\mu}\vec{H}) + \tfrac{1}{2}[\sqrt{\mu}\vec{H}, \nabla \ln\mu], \\ ik\sqrt{\epsilon\mu}\,(\sqrt{\mu}\vec{H}) &= \text{Curl}\,(\sqrt{\epsilon}\vec{E}) + \tfrac{1}{2}[\sqrt{\epsilon}\vec{E}, \nabla \ln\epsilon]. \end{aligned}\right\} \quad (4)$$

and having taken $\sqrt{\epsilon}\vec{E}$ and $\sqrt{\mu}\vec{H}$ in the capacity of the field vectors, we can convince ourselves of the the fact that the large quantity ϵ or μ is only included in the form $\sqrt{\epsilon\mu}$; however the quantities ϵ and μ taken separately are included in the form $\frac{1}{\epsilon}\nabla\epsilon$ and $\frac{1}{\mu}\nabla\mu$ (i.e., only their relative variations have an effect).

In order to compose a solution which yields an approximation of geometric optics we therefore assume that

$$\vec{E} = \frac{\vec{A}}{\sqrt{\epsilon}} e^{+\frac{i\psi}{q}}; \quad \vec{H} = \frac{\vec{B}}{\sqrt{\mu}} e^{+\frac{i\psi}{q}} \quad (5)$$

and making use of formula (4), we obtain the following equations:

$$\left.\begin{aligned} v\vec{A} + \frac{1}{k}[\nabla\psi, \vec{B}] &= -\frac{q}{ik}\left\{\text{Curl}\,\vec{B} + \frac{1}{2}[\vec{B}, \nabla \ln\mu]\right\} \\ \frac{1}{k}[\nabla\psi, \vec{A}] - v\vec{B} &= -\frac{q}{ik}\left\{\text{Curl}\,\vec{A} + \frac{1}{2}[\vec{A}, \nabla \ln\epsilon]\right\} \end{aligned}\right. \quad (6)$$

The solution for A and B is sought in the form of a power series of q:

$$\left.\begin{aligned} \vec{A} &= \vec{A}_o + q\vec{A}_1 + q^2\vec{A}_2 + \dots \\ \vec{B} &= \vec{B}_o + q\vec{B}_1 + q^2\vec{B}_2 + \dots \end{aligned}\right\} \quad (7)$$

Then we obtain the following system of equations:

$$v\vec{A}_o + \frac{1}{k}[\nabla\psi, \vec{B}_o] = 0; \quad \frac{1}{k}[\nabla\psi, \vec{A}_o] - v\vec{B}_o = 0, \quad (8)$$

$$\left.\begin{aligned} v\vec{A}_1 + \frac{1}{k}\left[\nabla\psi, \vec{B}_1\right] &= -\frac{1}{ik}\left\{\text{Curl}\,\vec{B}_o + \frac{1}{2}\left[\vec{B}_o, \nabla \ln\mu\right]\right\}; \\ \frac{1}{k}\left[\nabla\psi, \vec{A}_1\right] - v\vec{B}_1 &= \frac{-1}{ik}\left\{\text{Curl}\,\vec{A}_o + \frac{1}{2}\left[\vec{A}_o, \nabla \ln\epsilon\right]\right\}; \end{aligned}\right\} \quad (9)$$

$$\left.\begin{aligned} v\vec{A}_2 + \frac{1}{k}\left[\nabla\psi, \vec{B}_2\right] &= \frac{-1}{ik}\left\{\text{Curl}\,\vec{B}_1 + \frac{1}{2}\left[\vec{B}_1, \nabla \ln\mu\right]\right\}; \\ \frac{1}{k}\left[\nabla\psi, \vec{A}_2\right] - v\vec{B}_2 &= \frac{-1}{ik}\left\{\text{Curl}\,\vec{A}_1 + \frac{1}{2}\left[\vec{A}_1, \nabla \ln\epsilon\right]\right\} \end{aligned}\right\} \quad (10)$$

The zero-approximation Eqs. (8) yield the approximation of geometric optics. The condition for this solvability is the "eikonal equation"

$$(\nabla\psi)^2 = k^2 v^2, \quad (11)$$

from which the complex function $\psi(\underline{x}, y, \underline{z})$ must be determined. At the surface of the body the tangential components of the field in the body must coincide with the tangential components on its external surface. Since we assume that the field outside the body varies slowly, it follows that $\psi = 0$ on the surface of the body. From this it follows that the real and imaginary parts of ψ are proportional to one another: surfaces with equal phases and equal amplitudes inside the body (in this zero-approximation) coincide, and the normals to these surfaces at points lying on the surface of the body coincide with the normal to the surface of the body.

Thus it is true in this case that

$$\nabla\psi = -\hat{n}kv, \quad (12)$$

where $\hat{n}$ is the external normal to the surface of the body. Equations (8) provide the the relationship between the field vectors, and this relationship is the same as that for a plane wave; thus it follows that to the degree that we can limit ourselves to this approximation, the conclusion drawn in § 1 is valid. In order to find the boundaries of applicability for this derivation it is necessary to calculate the subsequent approximations. The corresponding calculations are made in the paper by S.M. Rytov which we have cited above. Making use of formula (34) of this paper and introducing the x and y axes in the tangent plane which are directed along the main cross sections of the surface of the body, we obtain the following condition (μ = const) instead of the boundary Condition (1), (2):

$$E_x = \sqrt{\frac{\mu}{\epsilon}} H_y \left\{ 1 + \frac{1}{ik\sqrt{\epsilon\mu}} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} + \frac{\partial \ln \sqrt{\epsilon}}{\partial z} \right) \right\} \qquad (13)$$

and a corresponding one for E_y. Here ρ_1 and ρ_2 are the main radii of curvature to the surface at the point being examined. From this expression it is evident that we will obtain a correction of the order of $\frac{d}{\rho}$ and $\frac{\partial \ln \sqrt{\epsilon}}{\partial z} d$ (d is the depth of penetration). When the main radii of curvature are equal the curvature does not yield any correction in this approximation; in addition, the correction associated with an inhomogeneity depends solely upon the variation of ϵ along the normal.

For a plane surface of a homogeneous body the first-order corrections are equal to zero, and in order to evaluate the errors in this case it is necessary to calculate the second approximation. Making use of Formula (21) in the paper by S.M. Rytov, we obtain the boundary condition in the following form for this case:

$$E_x = \sqrt{\frac{\mu}{\epsilon}} \left\{ H_y + \frac{-1}{4ik^2\epsilon\mu} \left(2\frac{\partial^2 H_y}{\partial x \partial y} + \frac{\partial^2 H_x}{\partial x^2} - \frac{\partial^2 H_y}{\partial y^2} \right) . \right\} \qquad (14)$$

Let us note that if ϵ and μ depend upon x and y, it follows that the corresponding corrections are also included in this approximation.

3. In deriving all of the formulas in this section we made the following postulate: the field on the external surface of the body varies slowly. In order to answer the second question which has been posed and in order to thus establish the limits of applicability of the boundary conditions (1) it is necessary to clarify when the above postulate is valid.

We shall at first suppose that the body has a large absorption (i.e., we shall assume that $\sqrt{\epsilon\mu}$ is complex and that im $\epsilon\mu$ is a large quantity). In that case we can assert that the condition of a slow field variation when the wave traverses the surface of the body is fulfilled at all distances from the source which are large in comparison to a wavelength inside the body and in comparison to the depth of penetration d inside the body. Even if the sources of the field are located on the very surface of the body, waves which are propagated in the body and which produce a rapidly varying field will be attenuated at such distances.

Thus the conditions for the applicability of the boundary conditions (1) for absorbing bodies will be the following. The depth of penetration into the body and the wavelength in it must be small in comparison to the wavelength in the surrounding space, in comparison to the distances from the sources of the field and in comparison to the radii of curvature of the surface of the body. Variations of ϵ and μ of the body at a distance equal to the wavelength in the body (or at a distance equal to the depth of penetration) are small.

In the case where ϵ and μ are both real and there is no absorption the situation is different, and the fulfillment of considerably more rigorous conditions is required in order for the boundary condition (1) to apply.

In fact, in this case even if the sources lie far away from the surface of the body (outside it), waves may be present in the body which travel not only from the surface into the interior of the body but also from inside the body into the space outside. For example, if our body is a plane-parallel plate and is irradiated by a plane wave, then a wave will exist in it which is reflected from its rear surface and which travels in the direction of the forward surface. Therefore the derivation of the boundary conditions (1) which was made above is inapplicable here.

In the case where a body with large values of ϵ and μ has a plane boundary and occupies an infinite half-space (the other half-space is a vacuum) there will be no such waves; however in this case the boundary conditions are applicable only in the case when the sources are at distances from the body which are large in comparison with the wave-length in the vacuum. However, if the source is located at the surface or close to the surface then, as we know, not only waves with a velocity c are propagated along the surface of the body, but also waves with a velocity $\frac{c}{\sqrt{\epsilon\mu}}$ which (from the upper side of the surface as well) create a rapidly varying field in the plane of the surface; thus in this case the assumption concerning the slow variations of the field on the external surface of the body (which we made in our derivation above) is untrue.

In conclusion let us provide the result of the solution of the problem involved in the reflection of a plane wave from an infinite homogeneous half-space ($\mu = 1$, ϵ is large); this solution is obtained by means of applying the approximate boundary conditions. A coefficient of reflection is obtained which is equal to the following expression:

$$R = -\frac{\sqrt{\frac{\epsilon}{\epsilon_0}}\cos\phi - 1}{\sqrt{\frac{\epsilon}{\epsilon_0}}\cos\phi + 1},$$

where ϕ is the angle of incidence. A comparison with the accurate Fresnel expression:

$$R = -\frac{\sqrt{\frac{\epsilon}{\epsilon_0}}\cos\phi - \sqrt{1 - \frac{\epsilon_0}{\epsilon}\sin^2\phi}}{\sqrt{\frac{\epsilon}{\epsilon_0}}\cos\phi + \sqrt{1 - \frac{\epsilon_0}{\epsilon}\sin^2\phi}}$$

shows that an error is obtained which is in complete agreement with our general derivations; this error applies for real values of ϵ and is of the order of $\frac{\epsilon_0}{\epsilon}\sin^2\phi$.